VALTION TEKNILLINEN TUTKIMUSLAITOS
STATENS TEKNISKA FORSKNINGSANSTALT
THE STATE INSTITUTE FOR TECHNICAL RESEARCH, FINLAND

JULKAISU 94 PUBLICATION

A THEORETICAL STUDY ON THE EFFECT OF GRAVITATION ON DRYING WITH SPECIAL REFERENCE TO CONCRETE

Two solutions of the diffusion-type equation with a gravitation term

S. E. PIHLAJAVAARA
MATTI A. RANTA

HELSINKI 1965

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FOREWORD

Since 1961, one of the authors, S. E. Pihlajavaara, has been engaged in a research project termed the "Drying of Concrete" [1]. The research needs a good deal of mathematical formulation and in particular the solutions. A diffusion type equation governing the drying, which is influenced by external force under certain boundary and initial conditions, presented in this report, is an example of this type of treatment. The task of finding solutions to the equation mentioned was entrusted to Dr. M. A. Ranta, the other author.

It is self-evident that this report makes no claim to be exhaustive. The report is only the authors' first and brief attempt to clarify the effect in question. The authors will therefore be especially interested in receiving additional information and constructive criticism on this subject.

Readers may like to know that, wherever physics (Chapter 1) in this report is incomprehensible the author is S. E. Pihlajavaara (S. E. P.), and that, where mathematics (Chapter 2) is incorrect the author is unquestionably M. A. Ranta. Numerical examples were calculated by O. Sarlin with aid of electronic computer ELLIOTT 803.

Both authors wrote their contribution into English direct. The final linquistic revision was made by Mrs. Lorna J. Sundström.

It is a pleasure to acknowledge thanks to Prof. Arvo Nykänen, Director of this laboratory, whose support made this report possible.

May, 1965

STATE INSTITUTE FOR TECHNICAL RESEARCH
Laboratory of Concrete Technology
Physics Section

S. E. Pihlajavaara

CONTENTS

1 PHYSICS

10 LIST OF PRINCIPAL SYMBOLS

C	moisture content (concentration of "free" water and water vapour in concrete) (kg/m^3)
C_0	initial moisture content (at the start of drying) (kg/m^3)
C_e	equilibrium moisture content (at the end of drying) (kg/m^3)
$\bar{C}$	average moisture content in the whole body (kg/m^3)
Fo	Fourier number = $kh/l^2 = \alpha t$
h	time
$\vec{i_g}$	unit vector in the direction of gravitation
J	flux of moisture (kg/m^2 s)
J_u	flux for relative moisture content u (m/s)
$J(x, t)$	dimensionless flux of moisture (See Eq. (12)')
K_g	general gravitational mobility (Appendix)
k	moisture conductivity (or diffusivity) (m^2/s)
k_g	gravitational "mobility" of moisture (m/s)
l	characteristic thickness (here the thickness of a slab)
N	= u (in Chapter 2 and 3)
$\bar{N}$	= $\bar{u}$ (in Chapter 2 and 3)
O ( )	order of magnitude
u	= $(C - C_e)/(C_0 - C_e)$ relative dimensionless moisture content at a point of the body (=N)
$\bar{u}$	= $(\bar{C} - C_e)/(C_0 - C_e)$ relative dimensionless average moisture content of the body (= $\bar{N}$)
t	= $h k_g/l$ dimensionless time variable
x	= z/l dimensionless coordinate
z	= coordinate in the direction of gravitation
α	= $k/k_g l$ dimensionless parameter

11 INTRODUCTION

When a moisture flow, such as drying, in a porous medium takes place in the presence of an external field of force, as that of gravitation, it is possible that the moving particles and especially the moving fluid in bulk are subjected to a directed movement superimposed upon their more or less random diffusive motions.

If the effect of gravitation can be neglected, the drying phenomenon, in which we are mainly interested, tends to produce a uniform distribution of moisture, independent of direction. Accordingly, if we adopt the simple diffusion theory with constant diffusivity to form a basis of our mathematical formulation of the phenomenon, solutions can usually be found relatively easily in numerous publications [2]. (Numerical solutions in the form of nomographs of many important cases, and especially of concrete, have also been presented by the author (S. E. P.) in papers for the forthcoming Symposium [3]).

If the effect of the gravitional field is significant, the situation is not so simple and the direction of drying can be of importance. For instance in the mathematical expressions for the moisture movement in granular porous media, such as soils, the effect of gravitation has been taken into account [4]. The significance of the gravitational term in the mathematical formulation concerning the movement of moisture in porous solids in general, has been an accepted fact in many publications [5] [6] [7] [8]. However, in the treatment of the moisture movement in fine-porous solids, especially in the theory of drying, diffusion type transfer without the gravitational effect has generally formed the basis of its mathematics.

Luikov says in his book [9] that moisture transfer in capillary-porous colloidal media consists of several random diffusion-like motions of water in the form of a liquid and vapour and it therefore obeys the diffusion equation (without a gravity term). In their book Luikov & Mikhaylov [10] treat only cases where "the dispersing medium, e.g. a porous body, contains bound matter in a vapour state, a liquid state, and a solid state — a three-phase system of bound matter", without introducing any gravity term. The meaning of the word "bound" seems not to be unambiguous, but regardless, the general opinion tacitly accepted seems to be that the smaller the pores are in a porous medium the more moisture is bound and the smaller the effect of gravitation. The adsorption (adhesion) theories strongly support this opinion. Krischer [11] does mention that the gravitational effect can be neglected already in the drying of a sand bed of average particle size 0.2 mm. Luikov [20] says that gravity can be neglected for pores smaller than 10^{-3} cm (0.01 mm) "with the accuracy of 6 per cent". In addition, the gravitational force is often considered to be of relatively minor significance in engineering [19]. Consequently the omission of the gravitational drift is quite understandable, as regards the drying of concrete containing mainly minor pores (diameter a. 1000 ··· 10 Å) and also because of a poor knowledge of the details of drying of concrete in general. It has been assumed more or less tacitly that the effect of gravitation is of minor importance and not worthwhile studying at this stage of development. The latter reasoning was also accepted by the author (S. E. P.) until the RILEM Symposium 1964 on Transfer of Water in Porous Media, where prof. S. Irmay (Faculty of Engineering, Israel Institute of Technology, Haifa, Israel) inquired of the author (S. E. P.) about the neglected gravity term in the mathematical formulation of mass transfer in the author's report [1]. The author (S. E. P.) then answered he believed that the gravitation is of minor importance in the drying of ordinary concretes. However, this discussion initiated this research on the effects of gravitation on the drying of concrete. The present state of the theoretical part of this study is presented in this report.

The governing differential equation adopted, and to be presented in the following paragraph as Equation (8)′, was the diffusion equation with a gravitational term, obtained by Th. de Coudres in an article, Ann. der Phys. 1894, which seems to be the first publication on the question. This is the opinion of Mason & Weawer of the University of Wisconsin in The Settling of Small Particles in a Fluid, Physical Review, March 1924 [12]. Mason & Weawer give solutions of Equation (8)′ for the cases where the fluid and movable particles to be examined are under sealed conditions. The "diffusion" in a gravitational field has also been investigated by Fürth in several articles during the years 1926 ··· 1935 [13] [14] [15]. Fürth indicates that Smoluchovski has presented the differential Equation (8)′ in an article, Ann. d. Physik 1915, p. 1105. The latest piece of information by Jost [16] and Bosworth [17] seems to indicate that relatively little improvement on the "diffusion" in a field of force has taken place during 1930 ··· 1960, if we do not take into account the advancement in the theory of ultracentrifugation [18], which, indeed, seems to be the best research method on gravity effects. However, those engaged in soil research have continuously been interested in the effect of gravitation.

A precursory examination of the preliminary experiments made in our laboratory indicates that the effect of gravitation on the drying of dense concrete seems to be measurable but that the individual variations in drying properties of different specimens, although of the same "population", may completely disturb the gravity effect.

12 MATHEMATICAL FORMULATION

In the following it is assumed that the main features of the macroscopic mathematical formulation presented by the various authors in their articles referred to above are also valid for moist concrete during drying. At any rate, it seems reasonable to start from there. Further research will verify if the theory adopted here needs revision. (As to the drying of concrete it seems reasonable to assume that the drying is isothermal [1]. It is self-evident that the gravitational force is taken to be independent of body dimensions [17]).

The total flux (or flux density) is (See Appendix)

$$\vec{J} = - k \operatorname{grad} C + k_g C \vec{i_g} \tag{1)'$$

where -grad C is the nonexternal or diffusive driving force, k the moisture conductivity, C the variable moisture content in the porous medium (concrete), and $k_g C \vec{i_g}$ the forced drift attributable to gravitation. k_g is the velocity of moisture due to gravitation and $\vec{i_g}$ the unit vector in the direction of gravitation. It seems reasonable to change the variable C into u through the substitution

$$u = \frac{C - C_e}{C_0 - C_e} \qquad (2)'$$

where C_0 is the initial moisture content and C_e the equilibrium moisture content of concrete, which is a hygroscopic material with certain equilibrium moisture contents. The substitution yields

$$\vec{J} = -(C_0 - C_e)\ k\ \text{grad}\ u + (C_0 - C_e)\ k_g\ u\ \vec{i_g} + k_g\ C_e\ \vec{i_g} \qquad (3)'$$

The last term to the right of equation is assumed to be insignificant, if moisture in equilibrium with the partial water vapour pressure in the surrounding air, C_e, is assumed to be immovable. It is evident that this assumption is better the smaller C_e is, or the nearer C_e is to the absolute zero moisture content. If the equilibrium moisture content C_e or part of it, is movable due to gravitation, its definition would appear to be incorrect. Some experiments seem to show, indeed, that concrete does not attain any real equilibrium state in long-term tests.

Equation (3)′ thus reduces to the form

$$\vec{J_u} = \frac{\vec{J}}{(C_0 - C_e)} = -\ k\ \text{grad}\ u + k_g\ u\ \vec{i_g} \qquad (4)'$$

As usual, the rate of change of u may be equated to the negative divergence of the flux

$$\partial u/\partial h = \text{div}\,(k\ \text{grad}\ u) - \text{div}\,(k_g\ u\ \vec{i_g}) \qquad (5)'$$

If we assume that k and k_g are constants, which they are not in general, we obtain

$$\partial u/\partial h = k\ \text{div grad}\ u - k_g\ \frac{\partial u}{\partial z}\ , \qquad (6)'$$

with z-axis selected to be in the direction of gravitation.

In order to reduce Equation (6)′ to a more convenient form, the following dimensionless substitutions are made

$$x = \frac{z}{l}\ , \quad \alpha = \frac{k}{k_g\ l}\ , \quad t = \frac{h\ k_g}{l}\ ; \quad (\alpha\ t = \text{Fo}, \quad \text{Fo} = \text{Fourier number}) \qquad (7)'$$

and if, in addition, only the one-dimensional case is considered, we obtain

$$\partial u/\partial t = \alpha\,\partial^2 u/\partial x^2 - \partial u/\partial x \qquad (8)'$$

In the mathematical treatment of the problem presented in the succeeding paragraph the following form of Equation (8)′ (u is denoted by N),

$$\partial N/\partial t = \alpha\,\partial^2 N/\partial x^2 - \partial N/\partial x \quad \text{or,}$$
$$\alpha N'' - N' - \dot{N} = 0\ , \qquad (9)'$$

will be used in accordance with the usual brief derivative notation.

The boundary conditions are fixed for two infinite slabs with one of the two faces free and the other impermeable. In the first case, denoted by I, the free face of the slab faces up and in the second case, denoted by II, the free face of the slab faces down. The thickness of the slabs is 1 and the x-coordinate is directed downward with x = 0 at the upper surface of the slab in both cases. The boundary conditions are accordingly, (See Paragraph 221):

$$\text{Case I} \quad x = 0, \quad N = 0 \qquad x = 1, \quad \alpha N' - N = 0 \qquad (10)'$$

$$\text{Case II} \quad x = 0 \quad \alpha N' - N = 0 \qquad x = 1, \quad N = 0 \qquad (11)'$$

The initial condition in both cases is assumed to be

$$N = 1, \quad \text{for} \quad t = 0$$

The equations are valid at least for quasi-isotropic and quasi-homogeneous media. Concrete is assumed to be such a medium, which means that concrete is assumed to be a macroscopically isotropic and homogeneous medium.

The cases presented above have been selected in accordance with experimental work started by the author (S. E. P.) in 1964.

The main objectives in the following mathematical treatment is to find for both the aforementioned cases:

a) Distribution of the moisture content or concentration,

$$N(x, t)$$

b) Average moisture content or concentration in the whole slab

$$\bar{N}(t) = \int_0^1 N(x, t)\, dx$$

c) Flux of moisture or concentration

$$J(x, t) = \alpha N' - N$$

In practical application it is worthwhile to notice that

$$J\left[\frac{M}{L^2 \cdot S}\right] = -k\left[\frac{L^2}{S}\right] \partial C\left[\frac{M}{L^3}\right] / \partial z[L] + k_g\left[\frac{L}{S}\right](C - C_e)\left[\frac{M}{L^3}\right]$$

$$= -k_g\left[\frac{L}{S}\right](C_0 - C_e)\left[\frac{M}{L^3}\right] J(x, t)\,[1] \qquad (12)'$$

if M, L, S are respectively mass, length, and time in consistent units.

13 REMARKS

131 Examination of parameter α and its order-of-magnitude

As stated in Paragraph 12

$$\alpha \equiv \frac{k}{k_g \, l}$$

This expression shows that α is a dimensionless parameter, dependent on diffusion, gravitation, and body dimension. Both k and k_g must be functions of pore size, pore continuity, viscosity and content of the movable fluid (moisture). It seems logical, therefore, to assume that they are mutually dependent. When k increases, k_g increases, and vice versa. Consequently, it seems reasonable also to believe that, for a certain medium and movable fluid, the ratio k/k_g is approximately constant, at least within a certain fluid or moisture content range.

Mason & Weawer examined the settling of small particles in a fluid and showed that diffusional and gravitational terms were

$$k' = \frac{RT}{N_A \, 6\pi\mu a} \; ; \quad k_g' = \frac{2 g \, \delta \, a^2}{9\mu} ,$$

where N_A is the Avogadro's number, R the gas constant, T the absolute temperature. μ is the coefficient of viscosity of the fluid, a the radius of particles, g acceleration of gravity, and δ the effective density of the immersed particle, (that is, its density minus the density of the fluid). In the present report the following analogy has been used

Settling particles $\rightarrow$ settling fluid (moisture)

Fluid $\rightarrow$ porous solid (concrete)

and k and k_g are mainly functions of pore size, pore continuity, viscosity, and moisture content as presented above; hence, parameters k and k_g cannot be specified in detail.

If we accept the analogy adopted here, the figures in the article of Mason & Weawer seem to indicate that for sealed ordinary concrete, even for relatively big l-values,

$$\alpha = O(10),$$

where the right hand term indicates "order of 10".

This is, of course, an assumption based on the opinion that the effect of gravitation on the drying of a fine-porous material, such as concrete, is small.

The sedimentation experiments in liquids described in the references of Paragraph 11 indicate that for practical l-values

$$\alpha = O(0.1 \cdots 1)$$

The cases $l = \infty$, 0 are practically unrealistic, consequently, $\alpha = 0, \infty$ are also practically unrealistic, besides which the ratio k/k_g is neither virtually 0 nor ∞. In an impermeable medium $k = 0$, but at the same time we must have $k_g = 0$.

132 Examination of the concentrations

It is evident that the moisture content or concentration at saturation, C_s, must be of great importance. For example, if the initial moisture concentration $C_0 = C_s$, no gravitational flow in concrete far from non-saturated or open boundaries is possible. This limits the application of the theory presented in this report.

133 Examination of gravitational velocity k_g and its order-of-magnitude

To simplify the mathematical treatment it is assumed that the gravitational velocity k_g is constant. This means, of course, that k_g can be considered to be approximately constant for a certain medium, in a certain temperature range, and in a certain moisture content range. It seems logical to believe that in the examination of the drying of a certain isothermal concrete the gravitational velocity depends only on the moisture content. The diminution of moisture content means an increase in the viscosity of water, the retained water being more fixed and having a higher concentration of soluble materials than the water already escaped. Further, the shrinkage attributable to the reduction in moisture content decreases the pore size and probably also the pore continuity, being an additional reason, beside the increase in viscosity, for the probable decrease of the gravitational velocity in concrete subjected to drying. (The reasoning presented above is "valid" also for moisture conductivity k, see [1]).

When planning experiments to determine numerical values for k_g it is necessary to take into account the aforementioned considerations and that mentioned in Paragraph 132. Consequently, it is preferable to make experiments under isothermal conditions, in a narrow moisture content range, and it is desirable that the initial moisture content, C_0, is not near the saturation moisture content, C_s.

If we accept the questionable assumption presented in Paragraph 131, that, for concrete α is of the order of 10, even for big l-values (here we fix $l = 1$ m), and we further assume that for concrete k has an average value of 10^{-10} m^2/s, we obtain for k_g the following order of magnitude values

$$k_g = 10^{-9} \text{ m/s} \approx 0.1 \text{ mm/24 h}$$

It is clear that, when a porous medium dries, a flowing fluid (water) transfers in the forms of liquid and vapour. The moisture content determines which type of flow, liquid or vapour, dominates. In fine-porous material, moisture is more or less bound, being under various forces of different origin (van der Waals, capillary, brownian, osmotic), and therefore a "free" flow is hardly possible, at any rate not in a macroscopical sense. In addition, the character of porous media is disorder rather than order, which must be taken into account in attempting to create model representation and behaviour. Consequently, it seems unfeasible to form a clear concept of the physical meaning of k_g. Nevertheless, k_g is an over-all gravitational parameter, which has a dimension of a velocity, and gives, as hoped, some kind of idea of the gravitational effect or gravitational sedimentation of moisture during drying.

134 On the mathematical model adopted

At present, there is no reliable evidence that the mathematical model adopted here sufficiently describes the effect of gravitation on the drying of concrete or any other fine-porous medium. Experiments will validate or invalidate the model. However, the physical aspects dominating here do not affect the mathematics involved, which can be of importance in other connections.

14 REFERENCES

1. Pihlajavaara, S.E., Notes on the drying of concrete. Helsinki 1963. [Valtion teknillinen tutkimuslaitos. Tiedotus. Sarja III - Rakennus 74 (State Institute for Technical Research, Report, Series III - Building 74)]. (In English).

2. Pihlajavaara, S.E., Introductory bibliography for research on drying of concrete. Second edition, Helsinki 1964. [State Institute for Technical Research. Laboratory of Concrete Technology. Duplicated]. (In Finnish and in English).

3. Pihlajavaara, S.E., Estimation of the drying of concrete. Paper prepared for and presented at the RILEM Symposium 1965 on Moisture Problems in Buildings. Helsinki August 1965.

4. Gardener, W.R., Mathematics of isothermal water conduction in unsaturated soils. Washington, D.C., 1958. [Highway Res. Board, Special Rep. 40].

5. Philip, J.R. & De Vries, D.A., Moisture movement in porous materials under temperature gradients. Transactions, American Geophysical Union, Vol. 38, No. 2. April 1957.

6. De Vries, D.A., Simultaneous transfer of heat and moisture in porous media. Transaction, Am. Geophys. Union, Vol. 39, No. 5, October 1958.

7. Philips, J.P., Physics of water movement in porous solids. Washington, D.C., 1958. [Highway Res. Board, Special Rep. 40].

8. De Vries, D.A., Transport of heat and moisture in porous media. A mathematical-physical treatment. Der Ingenieur, 1962:5 (In Dutch).

9. Lykow, A.V., Experimentalle und theoretische Grundlagen der Trocknung. VEB Verlag Technik, Berlin 1955, p. 140.

10. Lykow, A.V. & Mikhaylov, Y.A., Theory of energy and mass transfer. Prentice-Hall Inc. Englewood Cliffs, N.J., 1961.

11. Krischer, O., Die Wissenschaftlichen Grundlagen der Trocknungstechnik. Springer Verlag Berlin 1963.

12. Mason, M. & Weawer, W., The settling of small particles in a fluid. The Physical Review, 23 (1924):3.

13. Fürth, R., Über ein Problem der Diffusion in Schwerefelde. Zeitschrift für Physik, 40 (1926):5.

14. Fürth, R. & Zuber, B. & Sittle, K., Untersuchungen über Diffusion in Flüssigkeiten I - V. Z.f. Physik, Dd. 79, (1932):5 - 6.

15. Fürth, R., Sedimentation und Brownsche Bewegung in Schwerefeld. An article in Frank & Mises, Differentialgleichungen der Physik. Braunschweig 1935.

16. Jost, W., Diffusion in solids, liquids, gases. Academic Press. Inc. New York 1960.

17. Bosworth, R.C.L., Transport processes in applied chemistry. Sir Isaac Pitman & Sons, Ltd. Sydney 1956.

18. Fujita, H., Mathematical Theory of Sedimentation Analysis..Academic Press. New York and London 1962.

19. Dicke, R.H. & Roll, P.G. & Weber, J., Gravity experiments. Int. Science and Technology, 1964:5.

20. Lykow, A.V., Transporterscheinungen in kapillarporösen Körpern. Berlin 1958. Akademic Verlag, p. 17.

15 APPENDIX

The mathematical formulation presented in 12 could have been done in a more concise and general way as follows:

$$\vec{J} = -k \nabla C + K_g \vec{i_g} \qquad (1)'$$

where $K_g \vec{i_g}$ is the gravitational part of the total flux $\vec{J}$. Then combining the continuity (the principle of conservation of matter) requirement

$$\partial C / \partial h = - \nabla \cdot \vec{J} \qquad (2)'$$

Case I

The boundary conditions are

$$N = 0 \qquad \text{when } x = 0 \text{ , permeable surface}$$
$$\alpha N' - N = 0 \qquad \text{when } x = 1 \text{ , impermeable surface}$$

The eigenvalues are the positive roots of the equation

$$\tan\lambda = 2\alpha\lambda.$$

They are obtained, if α differs sufficiently from zero or from one-half, through the series developed in Appendix 3. If α only just exceeds one-half the first root must be determined by trial and error, for instance. The series for λ_n becomes

$$\lambda_n = \varphi_n - \frac{1}{2\alpha}\varphi_n^{-1} - \left(\frac{1}{4\alpha^2} - \frac{1}{24\alpha^3}\right)\varphi^{-3} - \left(\frac{1}{4\alpha^3} - \frac{1}{12\alpha^4} + \frac{1}{160\alpha^5}\right)\varphi_n^{-5}$$

where the abbreviation φ_n means

$$\left.\begin{aligned} \varphi_n &= (2n+1)\pi/2 \qquad \text{for} \qquad 0 < \alpha \leq 1/2 \\ \varphi_n &= (2n-1)\pi/2 \qquad \text{for} \qquad \alpha > 1/2 \end{aligned}\right\} , \; n = 1,2,3,\ldots$$

The error in λ_n is equal to that in ε_n, see formula (18) in Appendix 3, for large values of α it will be

$$|\Delta\lambda_n| = 0(\alpha^{-4}\varphi_n^{-7}) \qquad \text{for} \qquad \alpha \gg 1.$$

ω_n depends on λ_n according to the formula

$$\omega_n = (1 + 4\alpha^2\lambda_n^2)/4\alpha.$$

The concentration is

$$N(x,t) = \sum_{n=1}^{\infty} \frac{8\alpha^2\lambda_n}{1 + 4\alpha^2\lambda_n^2 - 2\alpha}\left(1 \pm \frac{(-1)^n 2e^{-1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right) e^{-\omega_n t + x/2\alpha} \sin\lambda_n x$$

The average concentration is

$$\bar{N}(t) = \sum_{n=1}^{\infty} \frac{8\alpha^2\lambda_n}{1 + 4\alpha^2\lambda_n^2 - 2\alpha}\left(1 \pm \frac{(-1)^n 2e^{-1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right) \frac{4\alpha^2\lambda_n}{1+4\alpha^2\lambda_n^2} e^{-\omega_n t}$$

And for the flux through the surface x = 0 we have

$$J(t) = \sum_{n=1}^{\infty} \frac{8\alpha^3\lambda_n^2}{1 + 4\alpha^2\lambda_n^2 - 2\alpha}\left(1 \pm \frac{(-1)^n 2e^{-1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right) e^{-\omega_n t}$$

In these formulae the + sign refers to $\alpha > 1/2$ and the - sign to $0 < \alpha \leq 1/2$

Case II

The boundary conditions are

$$\alpha N' - N = 0 \qquad \text{when } x = 0 \text{ , impermeable surface}$$
$$N = 0 \qquad \text{when } x = 1 \text{ , permeable surface}$$

The eigenvalues are the positive roots of the equation

$$\tan\lambda = -2\alpha\lambda$$

They are obtained for every α sufficiently apart from zero through the series developed in Appendix 3

$$\lambda_n = \varphi_n + \frac{1}{2\alpha}\varphi_n^{-1} - \left(\frac{1}{4\alpha^2} + \frac{1}{24\alpha^3}\right)\varphi_n^{-3} + \left(\frac{1}{4\alpha^3} + \frac{1}{12\alpha^4} + \frac{1}{160\alpha^5}\right)\varphi_n^{-5}$$

where the abbreviation φ_n means

$$\varphi_n = (2n-1)\pi/2 \ , \qquad n = 1,2,3,\ldots$$

The error in λ_n is less than the first omitted term $a_7\varphi_n^{-7}$ which for large values of α is of the order

$$|\triangle\lambda_n| = 0(\alpha^{-4}\lambda_n^{-7}) \qquad \text{for} \qquad \alpha \gg 1$$

ω_n depends on λ_n according to the formula

$$\omega_n = (1 + 4\alpha^2\lambda^2)/4\alpha$$

The concentration is

$$N(x,t) = \sum_{n=1}^{\infty} \frac{8\alpha^2\lambda_n}{1 + 4\alpha^2\lambda_n^2 + 2\alpha}\left(1 - \frac{(-1)^n 2e^{1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right) e^{-\omega_n t - (1-x)/2\alpha} \quad \sin\lambda_n(1-x)$$

The average concentration is

$$\bar{N}(t) = \sum_{n=1}^{\infty} \frac{8\alpha^2\lambda_n}{1 + 4\alpha^2\lambda_n^2 + 2\alpha}\left(1 - \frac{(-1)^n 2e^{1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right)\frac{4\alpha^2\lambda_n}{1 + 4\alpha^2\lambda_n^2} \quad e^{-\omega_n t}$$

And finally we have the flux through the surface x = 1,

$$J(t) = -\sum_{n=1}^{\infty} \frac{8\alpha^3\lambda_n^2}{1 + 4\alpha^2\lambda_n^2 + 2\alpha}\left(1 - \frac{(-1)^n 2e^{1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right) e^{-\omega_n t}$$

24 LITERATURE

Sneddon, I. N., Elements of partial differential equations, McGraw-Hill Book Company, Inc. New York 1957, pp. 327.

Churchill, R. V., Fourier series and boundary value problems, McGraw-Hill Book Company, Inc. New York 1941, pp. 206.

Wulich, B. S., Einführung in die Funktionalanalysis, Teile I, II, B. G. Teubner Verlagsgesellschaft, Leipzig 1961, Teil I, pp. 202, Teil II, pp. 171.

25 APPENDICES

Appendix 1

Contribution of eigenvalue $\lambda = 0$

In the two cases I and II the transcendental Equations (24) and (29) have a common root $\lambda = 0$ leading to $\omega = 1/4\alpha$. The characteristic Equation (18) has then a double root $r_1 = r_2 = -1/2\alpha$ and the corresponding solution and its derivative become

$$X(x) = e^{-x/2\alpha}(A + Bx)$$

$$X'(x) = e^{-x/2\alpha}\left[-\frac{1}{2\alpha}A + \left(1 - \frac{x}{2\alpha}\right)B\right]$$

The constants A and B are governed by the boundary conditions.

Case I

$$X(0) = A = 0$$

$$X'(1) = e^{-1/2\alpha}\left(1 - \frac{1}{2\alpha}\right)B = 0$$

The only possible solutions are $A = B = 0$.

Case II

$$X'(0) = -\frac{1}{2\alpha}A + B = 0$$

$$X(1) = e^{-1/2\alpha}(A + B) = 0$$

The only possible solutions are again $A = B = 0$.

The trivial solution $A = B = 0$ is the only possible one and thus there is no contribution to the series from $\lambda = 0$.

Appendix 2

a. The orthogonality of functions u_m and u_n

The vanishing of the scalar product (u_m , u_n) for $m \neq n$ was stated in Equation (37) without proof. This could be established naturally by the substitution of functions u_m and u_n from Equation (36) into Equation (37) and then performing the integration. However, an easier way is to utilizate directly the generating differential Equation (35) by rewritting it for the suffixes m and n

$$\begin{aligned} &a)\ \alpha (e^{x/\alpha} X')' + \omega_m e^{x/\alpha} X_m = 0 \\ &b)\ \alpha (e^{x/\alpha} X')' + \omega_n e^{x/\alpha} X_n = 0 \end{aligned} \qquad (1)$$

If the (1) a) is multiplied by X_n and (1) b) by X_m and the expressions so obtained are subtracted, then integration of the difference over the interval (0, 1) yields

$$(\omega_m - \omega_n)(u_m, u_n) = \alpha \int_0^1 dx \, [X_m (e^{x/\alpha} X_n')' - X_n (e^{x/\alpha} X_m')'] \qquad (2)$$

After integration by parts we arrive at

$$(\omega_m - \omega_n)(u_m, u_n) = \alpha \Big/_0^1 e^{x/\alpha} [X_m X_n' - X_m' X_n] - \alpha \int_0^1 dx \, e^{x/\alpha} [X_m' X_n' - X_n' X_m'] \qquad (3)$$

The first term vanishes because the product XX' is zero for $x = 0$ and $x = 1$ in case I as well as in II. In the second term the integrand vanishes identically. Consequently

$$(\omega_m - \omega_n)(u_m, u_n) = 0 \qquad (4)$$

Because the eigenvalues are all strictly distinct it is assured that

$$\omega_m - \omega_n \neq 0 \qquad \text{for} \qquad m \neq n \qquad (5)$$

from which we obtain the result stated before

$$(u_m, u_n) = 0 \qquad (6)$$

b. Determination of the square of the norm $\|u_n\|^2$

The simplest way is now straightforward integration of the expression

$$\| u_n \|^2 = \int_0^1 dx \sin^2 \lambda_n (x - \xi). \qquad (7)$$

Use of trigonometric formulae puts this into a form more suitable for integration

$$\|u\|^2 = \frac{1}{2}\int_0^1 dx\,[1 - \cos 2\lambda_n(x-\xi)] = \frac{1}{2}\Big/_2^1 \left[x - \frac{\sin 2\lambda_n(x-\xi)}{2\lambda_n}\right]$$

$$= \frac{1}{2}\left[1 - \frac{\sin 2\lambda_n(1-\xi) + \sin 2\lambda_n\xi}{2\lambda_n}\right] \tag{8}$$

This gives separately

for case I since $\xi = 0$ and $\tan\lambda_n = 2\alpha\lambda_n$

$$\|u_n\|^2 = \frac{1}{2}\left[1 - \frac{\sin 2\lambda_n}{2\lambda_n}\right] = \frac{1}{2}\left[1 - \frac{2\alpha}{1+4\alpha^2\lambda_n^2}\right] = \frac{1}{2}\left[1 - \frac{1}{2\omega_n}\right] \tag{9}$$

and for case II since $\xi = 1$ and $\tan\lambda_n = -2\alpha\lambda_n$

$$\|u_n\|^2 = \frac{1}{2}\left[1 - \frac{\sin 2\lambda_n}{2\lambda_n}\right] = \frac{1}{2}\left[1 + \frac{2\alpha}{1+4\alpha^2\lambda_n^2}\right] = \frac{1}{2}\left[1 + \frac{1}{2\omega_n}\right] \tag{10}$$

c. Determination of the scalar product $(u_n, e^{-x/2\alpha})$

According to the definition of the scalar product

$$(u_n, e^{-x/2\alpha}) = \int_0^1 dx\, e^{-x/2\alpha} \sin\lambda_n(x-\xi) \tag{11}$$

A change of variable $\eta = x - \xi$ in the integration gives

$$(u_n, e^{-x/2\alpha}) = e^{-\xi/2\alpha}\int_{-\xi}^{1-\xi} d\eta\, e^{-\eta/2\alpha}\sin\lambda_n\eta$$

$$= e^{-\xi/2\alpha}\Big/_{-\xi}^{1-\xi} e^{-\eta/2\alpha}\,\frac{(-1/2\alpha)\sin\lambda_n\eta - \lambda_n\cos\lambda_n\eta}{(-1/2\alpha)^2 + \lambda_n^2} \tag{12}$$

$$(u_n, e^{-x/2\alpha}) = \frac{4\alpha^2}{1+4\alpha^2\lambda_n^2}\left\{-e^{-1/2\alpha}\left[\frac{1}{2\alpha}\sin\lambda_n(1-\xi) + \lambda_n\cos\lambda_n(1-\xi)\right] - \left[\frac{1}{2\alpha}\sin\lambda_n\xi - \lambda_n\cos\lambda_n\xi\right]\right\} \tag{13}$$

Again separately we obtain

for case I with $\xi = 0$ and $\tan\lambda_n = 2\alpha\lambda_n$

$$(u_n, e^{-x/2\alpha}) = \frac{4\alpha^2}{1+4\alpha^2\lambda_n^2}\left\{\lambda_n - e^{-1/2\alpha}\, 2\lambda_n \cos\lambda_n\right\}$$

$$= \frac{4\alpha^2\lambda_n}{1+4\alpha^2\lambda_n^2}\left(1 \pm \frac{(-1)^n\, 2e^{-1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right); \quad \begin{matrix} + \text{ for } \alpha > 1/2 \\ - \text{ for } 0 < \alpha \leq 1/2 \end{matrix} \tag{14}$$

and for case II since $\xi = 1$ and $\tan\lambda_n = -2\alpha\lambda_n$

$$(u_n, e^{-x/2\alpha}) = \frac{4\alpha^2}{1+4\alpha^2\lambda_n^2}\left\{- e^{-1/2\alpha}\lambda_n + 2\lambda_n \cos\lambda_n\right\}$$

$$= -\frac{4\alpha^2\lambda_n}{1+4\alpha^2\lambda_n^2}\left(1 - \frac{(-1)^n\, 2e^{1/2\alpha}}{\sqrt{1+4\alpha^2\lambda_n^2}}\right) e^{-1/2\alpha} \tag{15}$$

d. Determination of the scalar product $(u_n, e^{x/2\alpha})$

By changing the sign of α in (13) we obtain immediately

$$(u_n, e^{x/2\alpha}) = \frac{4\alpha^2}{1+4\alpha^2\lambda_n^2}\left\{e^{1/2\alpha}\left[\frac{1}{2\alpha}\sin\lambda_n(1-\xi) - \lambda_n\cos\lambda_n(1-\xi)\right] + \frac{1}{2\alpha}\sin\lambda_n\xi + \lambda_n\cos\lambda_n\xi\right\} \tag{16}$$

from which it follows

for case I with $\xi = 0$ and $\tan\lambda_n = 2\alpha\lambda_n$

$$(u_n, e^{x/2\alpha}) = \frac{4\alpha^2\lambda_n}{1+4\alpha^2\lambda_n^2} \tag{17}$$

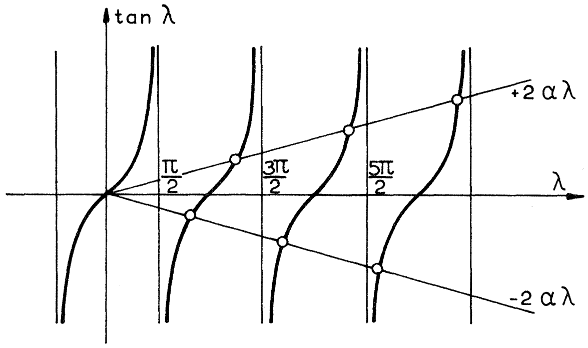
and for case II since $\xi = 1$ and $\tan\lambda_n = -2\alpha\lambda_n$

$$(u_n, e^{x/2\alpha}) = -\frac{4\alpha^2\lambda_n}{1+4\alpha^2\lambda_n^2}\, e^{1/2\alpha} \tag{18}$$

Appendix 3

An approximate series-method for the determination of the eigenvalues λ_n

As found the eigenvalues are the positive roots of the transcendental equations $\tan\lambda = \pm 2\alpha\lambda$. To get a better insight into these roots the graphs of the functions $\tan\lambda$ and $\pm 2\alpha\lambda$ have been sketched



From this figure it is apparent that the roots are quite close to the odd multiples of $\pi/2$. It leads us to put

$$\lambda_n = \varphi_n - \varepsilon_n \quad \text{for} \quad \text{case I} \qquad (1)$$

$$, \quad \text{where} \quad \lim_{n \to \infty} \varepsilon_n = 0$$

$$\lambda_n = \varphi_n + \varepsilon_n \quad \text{for} \quad \text{case II} \qquad (2)$$

In case I there exist two possibilities for φ_n depending on the magnitude of α, namely

$$\varphi_n = (2n - 1)\pi/2 \quad \text{for} \quad \alpha > 1/2 \qquad (3)$$

$$\varphi_n = (2n + 1)\pi/2 \quad \text{for} \quad 0 < \alpha \leq 1/2 \qquad (4)$$

while in case II for every $\alpha > 0$

$$\varphi_n = (2n - 1)\pi/2. \qquad (5)$$

There is no doubt that $\varepsilon_n = \varepsilon(\varphi_n, \alpha)$ is different in both of the cases I and II. Nevertheless they can be treated together if we substitute $\lambda_n = \varphi_n \mp \varepsilon_n$ into the equation $\tan\lambda_n = \pm 2\alpha\lambda_n$. The upper signs refer to case I and the lower signs to case II. After some manipulation and dropping the suffix n for brevity we obtain

$$\mp \varepsilon \tan\varepsilon + \varphi \tan\varepsilon - 1/2\alpha = 0. \qquad (6)$$

Appendix 3

The function $\tan\varepsilon$ can expanded into the series

$$\tan\varepsilon = \varepsilon + \frac{1}{3}\varepsilon^3 + \frac{2}{15}\varepsilon^5 + \cdots$$

the substitution of which into (6) yields

$$\varphi\varepsilon \mp \varepsilon^2 + \frac{1}{3}\varphi\varepsilon^3 \mp \frac{1}{3}\varepsilon^4 + \frac{2}{15}\varphi\varepsilon^5 \mp \frac{2}{15}\varepsilon^6 + \cdots = 1/2\alpha \tag{7}$$

For the solution of ε let us assume it can be expressed as a series of descending powers of φ

$$\varepsilon = a_1\varphi^{-1} + a_2\varphi^{-2} + a_3\varphi^{-3} + a_4\varphi^{-4} + a_5\varphi^{-5} + \cdots \tag{8}$$

and expand into powers as follows:

$$\left\{\begin{aligned} \varepsilon^2 &= a_1^2\varphi^{-2} + 2a_1a_2\varphi^{-3} + (a_2^2 + 2a_1a_3)\varphi^{-4} + \cdots \\ \varepsilon^3 &= a_1^3\varphi^{-3} + 3a_1^3a_2\varphi^{-4} + 3(a_1a_2^2 + a_1^2a_3)\varphi^{-5} + \cdots \\ \varepsilon^4 &= a_1^4\varphi^{-4} + \cdots \\ \varepsilon^5 &= a_1^5\varphi^{-5} + \cdots \end{aligned}\right. \tag{9}$$

After (8) and (9) have been substituted into (7) comparison of the coefficients of equal powers of φ leads to equations

$$\begin{aligned} &a_1 && = 1/2\alpha \\ &a_2 && = 0 \\ &a_3 \mp a_1^2 + \frac{1}{3}a_1^3 && = 0 \\ &a_4 \mp 2a_1a_2 + a_1^2a_2 && = 0 \\ &a_5 \mp (a_2^2 + 2a_1a_3) + (a_1a_2^2 + a_1^2a_3) \mp \frac{1}{3}a_1^4 + \frac{2}{15}a_1^5 && = 0 \end{aligned} \tag{10}$$

from which the coefficients can be determined as a function of coefficients of lower order:

$$\begin{aligned} a_1 &= 1/2\alpha \\ a_2 &= 0 \\ a_3 &= \pm a_1^2 - \frac{1}{3}a_1^3 \\ a_4 &= 0 \\ a_5 &= \pm 2a_1a_3 - a_1^2a_3 \pm \frac{1}{3}a_1^4 - \frac{2}{15}a_1^5 \end{aligned} \tag{11}$$

In terms of α the non-zero coefficients become

$$a_1 = 1/2\alpha \tag{12}$$

$$a_3 = \pm\, 1/4\alpha^2 - 1/24\alpha^3 \tag{13}$$

$$a_5 = 1/4\alpha^3 \mp 1/12\alpha^4 + 1/160\alpha^5 \tag{14}$$

The general law for formation of the numerical factors in the n-th coefficient a_n is unclear by reason of the tan-series. But on the ground of Equations (11) ··· (14) we can obviously deduce

$$a_{2k} = 0 \tag{15}$$

$$|a_{2k-1}| < \sum_{j=k}^{2k-1} \alpha^{-j} = \frac{\alpha^{-k} - \alpha^{-2k}}{1 - \alpha^{-1}}\,, \quad k = 1,2,3,\ldots \tag{16}$$

The error introduced by the omission of the terms of higher order than the 5-th, when calculating ε_n by the series (8), is

$$|\triangle \varepsilon_n| < \frac{1}{1 - \alpha^{-1}} \sum_{k=4}^{\infty} \left\{ \alpha^{-k}\, \varphi_n^{-(2k-1)} - \alpha^{-2k}\, \varphi_n^{-(2k-1)} \right\} \tag{17}$$

or in a more compact form

$$|\triangle \varepsilon_n| < \frac{\alpha^{-4}\varphi_n^{-7}}{1 - \alpha^{-1}} \left\{ \frac{1}{1 - \alpha^{-1}\varphi_n^{-2}} - \frac{\alpha^{-4}}{1 - \alpha^{-2}\varphi_n^{-2}} \right\}. \tag{18}$$

If we introduce a restriction to large values of α only, the error (18) is of the order of magnitude

$$|\triangle \varepsilon_n| = 0\,(\alpha^{-4}\, \varphi_n^{-7}) \qquad \text{for} \qquad \alpha \gg 1 \tag{19}$$

By formulae (1) ··· (5), (8), and (12) ··· (14) the eigenvalues λ_n can be determined approximately and by Equations (18) or (19) the error can be estimated. For small values of α and n the error $|\triangle \varepsilon_n|$, Equation (19), could be too great and therefore at least some of the smallest eigenvalues must be determined by some other method, e.g. by trial and error.

3 NUMERICAL EXAMPLES

3 NUMERICAL EXAMPLES

Numerical examples have been calculated using the formulae presented in Paragraph 23 ("Summary of formulae") with the aid of an electronic computer, ELLIOT 803 A.

For the evaluation of the eigenvalue λ_n we have from Paragraph 23 the formula

$$\lambda_n = \varphi_n + A'\varphi_n^{-1} + B'\varphi_n^{-3} + C'\varphi_n^{-5} + \triangle\lambda_n \qquad (1)''$$

where $\triangle\lambda_n$ is the residual error the upper absolute limit of which is given in Appendix 3 (Formulae (18) and (19))

$$|\triangle\lambda_n| = |\triangle\varepsilon_n| = 0(\alpha^{-4}\varphi_n^{-7}) \qquad (2)''$$

For $\alpha \neq 1$, the first term omitted is $\triangle\lambda_n$ when evaluating the eigenvalues λ_n (n = 1,2,3), while for λ_n (n > 3) we omit $C'\varphi_n^{-5}$.

For $\alpha = 1$, using formula (18) of Appendix 3 we arrive at the limit

$$\lim_{\alpha \to 1} |\triangle\lambda_n| < \varphi_n^{-7}\left\{\frac{4 - 3\varphi_n^{-2}}{(1 - \varphi_n^{-2})^2}\right\} \qquad (3)''$$

From this we obtain for different values of n:

	$\|\triangle\lambda_n\| <$
n = 1	0.333607
n = 2	0.000082
n = 3	0.000002

Since errors of such a magnitude are unpermissible a special iteration program was adopted to the evaluation of λ_n for $\alpha = 1$. The error in λ_n computed by this program is less that 10^{-7} for $n < 4$. The corrections for the two first λ_n were taken into account for the computation of the final results. The values of $\triangle\lambda_1$ were checked also for $\alpha = 5$ and $\alpha = 10$.

For the sum of the upper limits of the absolute errors due to the neglected terms in this series for eigenvalues we obtained:

$3.7 \cdot 10^{-6}$	for	$\alpha = 1$
$21.5 \cdot 10^{-6}$	for	$\alpha = 5$
$1.4 \cdot 10^{-6}$	for	$\alpha = 10$
$0.3 \cdot 10^{-6}$	for	$\alpha = 20$
$0.0 \cdot 10^{-6}$	for	$\alpha = 50$

In studying the effect of errors in eigenvalues on the final results, the complicated expressions derived from the formulae in Paragraph 23 can be replaced by the numerical values obtained for the cases studied:

$$\triangle N(x,t)/\triangle\lambda \le 1.05$$

$$\triangle \bar{N}(t)/\triangle\lambda \le 1.00$$

$$\triangle J(t)/\triangle\lambda \le 45$$

$\triangle J/\triangle\lambda$ possesses large values for large α only, and accordingly the final errors are

$$< 5 \cdot 10^{-6} \quad \text{for} \quad \alpha \neq 5$$

$$< 5 \cdot 10^{-5} \quad \text{for} \quad \alpha = 5$$

The convergence of the series-solutions for the moisture distribution, average moisture, and the flux were tested by following the series of $N(x,t)$ for every pair of α, t when the convergence was at the lowest in case I as well as II. The corresponding x-coordinate is denoted by x_r. In case I $x_r = 1$ and in case II $x_r = 0$. For every pair of values α,t the last term of every series taken corresponds to the eigenvalue λ_n, which gives at the first attempt less than 10^{-5} for the change in $N(x_r, t)$. Exceptions are the extra points of the $N(x,t)$-distribution at small values of t and at values of x close to the impermeable surface. In these cases the convergence was checked separately.

In accordance with the above mentioned, the results in all cases are correct to at least four decimals.

The α-values have been selected using the knowledge that the greatest α-value ($\alpha = 50$) corresponds to the case in which the effect of gravitation is insignificant, and using the assumption that the smallest α-value ($\alpha = 1$) gives the greatest possible gravitational effect for concrete. The t-values used have been selected so that the different stages of the drying process can be followed with "sufficient" accuracy.

The results of the examples have been presented as 4-decimal Tables 1 ··· 7 and with aid of the numerical tables Figures 1 ··· 13 were drawn for both cases, I and II, consisting of curves $N(x,t)$ as a function of $\alpha = 1, 5, 10, 20, 50$ (Figures 1 ··· 10), $N(t,\alpha)$ (Figure 11), and $J(t,\alpha)$ (Figure 12). In addition a family of curves for $\bar{N}(Fo,\alpha)$ has been presented in order to clarify the relation between ordinary diffusion and the gravitational effect (Figure 13). The corresponding relation could have been obtained with aid of a curve family $J(t)/\alpha$ using Fo as an abscissa.

TABLE 1

$N(x,t)$, when $\alpha = 1$, Case I

x	t=0.0001	0.0003	0.001	0.003	0.01	0.03	0.05	0.1	0.2	0.3	0.5	1.0	2.0	3.0
0.01	.5180	.3134	.1728											
0.02	.8413	.5816	.3387											
0.05	.9996	.9577	.7298											
0.075	1.0001	.9978	.9029											
0.1	1.0000	1.0000	.9734	.7933	.4964	.2834	.2120	.1418	.1004	.0832	.0600	.0269	.0054	.0011
0.125	1.0001	1.0000	.9945											
0.15	1.0001	1.0000	.9991											
0.2	1.0000	1.0000	1.0000	.9892	.8264	.5438	.4211	.2903	.2091	.1736	.1253	.0561	.0112	.0022
0.3	1.0000	1.0000	1.0000	.9999	.9607	.7450	.6064	.4387	.3244	.2705	.1954	.0874	.0175	.0035
0.4	1.0000	1.0000	1.0000	1.0000	.9943	.8764	.7559	.5814	.4447	.3728	.2695	.1206	.0241	.0048
0.5	1.0000	1.0000	1.0000	1.0000	.9995	.9504	.8677	.7153	.5686	.4793	.3469	.1552	.0311	.0062
0.6	1.0000	1.0000	1.0000	1.0000	1.0001	.9898	.9496	.8401	.6945	.5888	.4266	.1909	.0382	.0076
0.7	1.0000	1.0000	1.0000	1.0000	1.0015	1.0176	1.0156	.9583	.8213	.6998	.5074	.2270	.0454	.0091
0.8	1.0000	1.0000	1.0000	1.0002	1.0093	1.0531	1.0817	1.0744	.9477	.8107	.5881	.2631	.0527	.0105
0.9	1.0000	1.0000	1.0004	1.0070	1.0393	1.1128	1.1632	1.1938	1.0724	.9197	.6674	.2986	.0598	.0120
1.0	1.0113	1.0197	1.0362	1.0633	1.1179	1.2108	1.2730	1.3210	1.1940	1.0248	.7438	.3328	.0666	.0133
Case II														
0.0	.9888	.9806	.9648	.9397	.8921	.8190	.7698	.6622	.4699	.3280	.1590	.0260		
0.1	1.0000	1.0000	.9996	.9926	.9595	.8904	.8388	.7206	.5104	.3562	.1726	.0282		
0.2	1.0000	1.0000	1.0000	.9997	.9892	.9398	.8896	.7606	.5361	.3738	.1812	.0296		
0.3	1.0000	1.0000	1.0000	1.0000	.9981	.9683	.9189	.7782	.5443	.3791	.1837	.0300		
0.4	1.0000	1.0000	1.0000	1.0000	.9998	.9772	.9224	.7691	.5328	.3706	.1795	.0293		
0.5	1.0000	1.0000	1.0000	1.0000	.9997	.9635	.8936	.7291	.4997	.3471	.1681	.0275		
0.6	1.0000	1.0000	1.0000	1.0000	.9962	.9151	.8238	.6546	.4440	.3079	.1491	.0244		
0.7	1.0000	1.0000	1.0000	.9999	.9709	.8105	.7029	.5435	.3651	.2529	.1224	.0200		
0.8	1.0000	1.0000	1.0000	.9911	.8579	.6264	.5237	.3954	.2636	.1824	.0883	.0144		
0.85	.9998	1.0000	.9992											
0.875	.9998	1.0000	.9951											
0.9	1.0000	1.0000	.9759	.8130	.5444	.3515	.2862	.2126	.1410	.0975	.0472	.0077		
0.925	1.0004	.9979	.9099											
0.95	1.0001	.9597	.7430											
0.98	.8449	.5899	.3518											
0.99	.5224	.3203	.1811											

TABLE 2

N(x,t) , when $\alpha = 5$, Case I

x	t=0.0001	0.0003	0.001	0.003	0.01	0.03	0.05	0.1	0.2	0.3	0.5
0.01	.2473	.1440	.0787								
0.02	.4717	.2836	.1568								
0.05	.8857	.6369	.3798								
0.075	.9821	.8278	.5433								
0.1	.9984	.9314	.6795	.4307	.2408	.1386	.1053	.0592	.0190	.0061	.0006
0.125	.9998	.9772	.7861								
0.15	1.0001	.9937	.8644								
0.2	1.0000	.9997	.9536	.7468	.4626	.2754	.2101	.1182	.0379	.0122	.0013
0.3	1.0000	1.0000	.9972	.9142	.6474	.4060	.3120	.1757	.0564	.0181	.0019
0.4	1.0000	1.0000	.9999	.9782	.7871	.5267	.4084	.2303	.0739	.0237	.0024
0.5	1.0000	1.0000	1.0000	.9959	.8833	.6345	.4971	.2808	.0901	.0289	.0030
0.6	1.0000	1.0000	1.0000	.9997	.9447	.7276	.5758	.3258	.1046	.0336	.0035
0.7	1.0000	1.0000	1.0000	1.0011	.9826	.8048	.6429	.3643	.1169	.0375	.0039
0.8	1.0000	1.0000	1.0003	1.0042	1.0076	.8657	.6966	.3952	.1268	.0407	.0042
0.9	1.0000	1.0003	1.0033	1.0121	1.0278	.9099	.7357	.4177	.1340	.0430	.0044
1.0	1.0050	1.0088	1.0160	1.0279	1.0480	.9372	.7590	.4311	.1383	.0444	.0046

Case II

x	t=0.0001	0.0003	0.001	0.003	0.01	0.03	0.05	0.1	0.2	0.3	0.5
0.0	.9950	.9913	.9841	.9726	.9477	.7954	.6171	.3170	.0833	.0219	.0015
0.1	1.0000	.9997	.9966	.9879	.9628	.8020	.6213	.3191	.0838	.0220	.0015
0.2	1.0000	1.0000	.9996	.9957	.9690	.7893	.6092	.3127	.0821	.0216	.0015
0.3	1.0000	1.0000	1.0000	.9987	.9631	.7565	.5805	.2977	.0782	.0205	.0014
0.4	1.0000	1.0000	1.0000	.9992	.9392	.7032	.5357	.2744	.0721	.0189	.0013
0.5	1.0000	1.0000	1.0000	.9963	.8885	.6291	.4754	.2432	.0639	.0168	.0012
0.6	1.0000	1.0000	.9999	.9799	.8007	.5348	.4010	.2048	.0538	.0141	.0010
0.7	1.0000	1.0000	.9974	.9192	.6668	.4216	.3139	.1602	.0421	.0111	.0008
0.8	1.0000	.9997	.9554	.7567	.4832	.2922	.2164	.1103	.0290	.0076	.0005
0.85	1.0001	.9939	.8684								
0.875	.9998	.9778	.7913								
0.9	.9984	.9328	.6859	.4419	.2557	.1502	.1108	.0565	.0148	.0039	.0003
0.925	.9823	.8304	.5501								
0.95	.8868	.6405	.3860								
0.98	.4738	.2864	.1602								
0.99	.2488	.1457	.0806								

TABLE 3

N(x,t) , when $\alpha = 10$, Case I

x	t=0.0001	0.0003	0.001	0.003	0.01	0.02	0.03	0.05	0.1	0.3
0.01	.1757	.1023	.0559							
0.02	.3434	.2030	.1116							
0.05	.7362	.4801	.2745							
0.075	.9061	.6658	.4019							
0.1	.9745	.8023	.5181	.3135	.1733	.1217	.0944	.0586	.0179	.0002
0.125	.9949	.8928	.6209							
0.15	.9992	.9468	.7090							
0.2	1.0000	.9901	.8411	.5816	.3399	.2413	.1875	.1165	.0356	.0003
0.3	1.0000	.9999	.9656	.7760	.4921	.3556	.2770	.1721	.0527	.0005
0.4	1.0000	1.0000	.9952	.8956	.6247	.4615	.3607	.2243	.0686	.0006
0.5	1.0000	1.0000	.9996	.9581	.7348	.5565	.4366	.2716	.0831	.0007
0.6	1.0000	1.0000	1.0000	.9863	.8220	.6385	.5027	.3129	.0957	.0008
0.7	1.0000	1.0000	1.0002	.9982	.8878	.7059	.5576	.3472	.1062	.0009
0.8	1.0000	1.0000	1.0010	1.0046	.9349	.7573	.5997	.3736	.1143	.0010
0.9	1.0000	1.0007	1.0040	1.0109	.9658	.7918	.6281	.3914	.1197	.0010
1.0	1.0036	1.0062	1.0113	1.0196	.9825	.8089	.6420	.4001	.1224	.0011

Case II

x	t=0.0001	0.0003	0.001	0.003	0.01	0.02	0.03	0.05	0.1	0.3
0.0	.9962	.9938	.9888	.9805	.9170	.7370	.5730	.3431	.0950	.0006
0.1	1.0000	.9993	.9960	.9886	.9183	.7353	.5714	.3421	.0947	.0006
0.2	1.0000	1.0000	.9990	.9932	.9032	.7155	.5552	.3323	.0920	.0005
0.3	1.0000	1.0000	.9998	.9932	.8696	.6775	.5245	.3138	.0869	.0005
0.4	1.0000	1.0000	1.0000	.9850	.8148	.6219	.4801	.2871	.0795	.0005
0.5	1.0000	1.0000	.9996	.9594	.7362	.5494	.4228	.2527	.0700	.0004
0.6	1.0000	1.0000	.9954	.8994	.6322	.4615	.3540	.2115	.0585	.0003
0.7	1.0000	.9999	.9666	.7826	.5028	.3598	.2752	.1644	.0455	.0003
0.8	1.0000	.9903	.8443	.5899	.3505	.2470	.1885	.1125	.0311	.0002
0.875	.9941	.8941								
0.9	.9747	.8043	.5229	.3203	.1805	.1259	.0960	.0573	.0159	.0001
0.925	.9079	.6684								
0.95	.7375	.4827	.2781							
0.98	.3447	.2045	.1134							
0.99	.1765	.1032	.0568							

TABLE 4

N(x,t) , when α = 20 , Case I

x	t=0.0001	0.0003	0.001	0.003	0.005	0.01	0.02	0.03	0.05	0.1
0.05	.5702	.3511								
0.1	.8859	.6378	.3814	.2253	.1751	.1228	.0743	.0458	.0174	.0016
0.15	.9821	.8285								
0.2	.9984	.9318	.6811	.4336	.3426	.2428	.1472	.0907	.0345	.0031
0.25	.9998	.9774								
0.3	1.0000	.9938	.8654	.6109	.4948	.3567	.2169	.1337	.0508	.0045
0.4	1.0000	.9997	.9541	.7499	.6267	.4616	.2817	.1736	.0660	.0059
0.5	1.0000	1.0000	.9875	.8503	.7353	.5549	.3399	.2096	.0797	.0071
0.6	1.0000	1.0000	.9974	.9175	.8203	.6345	.3903	.2406	.0915	.0082
0.7	1.0000	1.0000	1.0001	.9594	.8834	.6989	.4314	.2660	.1011	.0090
0.8	1.0000	1.0002	1.0016	.9842	.9270	.7468	.4623	.2851	.1084	.0097
0.9	1.0002	1.0011	1.0039	.9983	.9538	.7775	.4822	.2974	.1130	.0101
1.0	1.0025	1.0044	1.0080	1.0059	.9658	.7904	.4905	.3025	.1150	.0102

Case II

x	t=0.0001	0.0003	0.001	0.003	0.005	0.01	0.02	0.03	0.05	0.1
0.0	.9975	.9956	.9920	.9787	.9330	.7545	.4588	.2773	.1013	.0082
0.1	.9998	.9989	.9960	.9800	.9301	.7492	.4553	.2752	.1005	.0081
0.2	1.0000	.9999	.9983	.9729	.9111	.7259	.4404	.2662	.0972	.0078
0.3	1.0000	1.0000	.9990	.9536	.8742	.6847	.4143	.2504	.0915	.0074
0.4	1.0000	1.0000	.9972	.9158	.8167	.6262	.3776	.2282	.0834	.0067
0.5	1.0000	1.0000	.9877	.8518	.7360	.5514	.3312	.2002	.0731	.0059
0.6	1.0000	.9997	.9549	.7537	.6304	.4616	.2763	.1669	.0610	.0049
0.7	1.0000	.9939	.8674	.6161	.5002	.3589	.2140	.1293	.0472	.0038
0.75	.9998	.9777								
0.8	.9984	.9324	.6843	.4390	.3479	.2456	.1461	.0883	.0323	.0026
0.85	.9823	.8297								
0.9	.8864	.6396	.3845	.2291	.1787	.1249	.0742	.0448	.0164	.0013
0.95	.5713	.3527								

TABLE 5

N(x,t) , when $\alpha = 50$, Case I

x	t=0.0001	0.0003	0.001	0.003	0.005	0.01	0.02	0.03
0.05	.3826							
0.1	.6824	.4357	.2474	.1439	.1080	.0581	.0171	.0050
0.15	.8662							
0.2	.9544	.7513	.4719	.2832	.2133	.1150	.0338	.0099
0.25	.9876							
0.3	.9973	.9165	.6562	.4134	.3135	.1691	.0497	.0146
0.4	.9999	.9790	.7934	.5309	.4058	.2192	.0645	.0190
0.5	1.0000	.9961	.8859	.6328	.4881	.2641	.0777	.0228
0.6	1.0000	.9995	.9425	.7173	.5584	.3026	.0890	.0262
0.7	1.0000	1.0001	.9741	.7835	.6151	.3337	.0982	.0289
0.8	1.0000	1.0004	.9904	.8312	.6569	.3568	.1049	.0309
0.9	1.0003	1.0012	.9983	.8604	.6829	.3712	.1092	.0321
1.0	1.0016	1.0028	1.0019	.8713	.6926	.3765	.1107	.0326

Case II

x	t=0.0001	0.0003	0.001	0.003	0.005	0.01	0.02	0.03
0.0	.9984	.9972	.9919	.8572	.6784	.3651	.1053	.0303
0.1	.9997	.9988	.9918	.8497	.6715	.3613	.1042	.0300
0.2	1.0000	.9996	.9865	.8236	.6481	.3485	.1005	.0290
0.3	1.0000	.9998	.9721	.7787	.6088	.3270	.0943	.0272
0.4	1.0000	.9994	.9419	.7149	.5544	.2974	.0857	.0247
0.5	1.0000	.9961	.8864	.6322	.4859	.2603	.0750	.0216
0.6	.9999	.9792	.7948	.5317	.4050	.2167	.0625	.0180
0.7	.9973	.9170	.6582	.4150	.3137	.1676	.0483	.0139
0.75	.9876							
0.8	.9546	.7523	.4739	.2849	.2140	.1142	.0329	.0095
0.85	.8666							
0.9	.6830	.4369	.2489	.1451	.1085	.0579	.0167	.0048
0.95	.3832							

TABLE 6 : $\bar{N}$ (t)

	α=1		α=5		α=10		α=20		α=50	
t	I	II	I	II	I	II	I	II	I	II
0.0001	.9888	.9887	.9748	.9747	.9644	.9643	.9496	.9495	.9203	.9202
0.0003	.9806	.9803	.9564	.9561	.9383	.9380	.9127	.9124	.8620	.8617
0.001	.9648	.9638	.9207	.9197	.8877	.8867	.8409	.8399	.7482	.7472
0.003	.9397	.9367	.8633	.8603	.8061	.8031	.7251	.7221	.5644	.5617
0.005							.6456	.6407	.4397	.4358
0.01	.8921	.8821	.7526	.7426	.6480	.6383	.5002	.4916	.2383	.2338
0.02					.5044	.4874	.3077	.2965	.0701	.0674
0.03	.8191	.7891	.5768	.5490	.3973	.3763	.1897	.1792	.0206	.0194
0.05	.7716	.7217	.4571	.4183	.2473	.2250	.0721	.0655		
0.1	.6895	.5919	.2587	.2142	.0756	.0623	.0064	.0053		
0.2	.5778	.4088	.0830	.0563						
0.3	.4909	.2843	.0266	.0148	.0007	.0004				
0.5	.3557	.1377	.0027	.0010						
1	.1592	.0225								
2	.0319									
3	.0064	.0000								

TABLE 7 : J (t)

	α=1		α=5		α=10		α=20		α=50	
t	I	II	I	II	I	II	I	II	I	II
0.0001	55.91	56.90	125.65	126.64	177.9	178.9	251.8	252.8	398.4	399.4
0.0003	32.07	33.07	72.34	73.34	102.5	103.5	145.2	146.2	229.8	230.8
0.001	17.35	18.34	39.40	40.40	55.92	56.92	79.29	80.29	125.7	126.7
0.003	9.808	10.808	22.54	23.54	32.08	33.08	45.58	46.56	72.29	73.02
0.005							35.23	36.13	54.16	54.55
0.01	5.156	6.156	12.12	13.12	17.39	18.29	24.61	25.17	29.16	29.08
0.02					12.17	12.72	14.89	14.94	8.576	8.385
0.03	2.782	3.782	6.901	7.630	9.435	9.688	9.175	9.024	2.523	2.417
0.05	2.057	3.050	5.234	5.622	5.858	5.781	3.488	3.296		
0.1	1.361	2.252	2.941	2.864	1.792	1.600	.3109	.2658		
0.2	.9585	1.4912	.9436	.7523						
0.3	.7930	1.0309	.3028	.1976	.01569	.00939				
0.5	.5722	.4989	.03118	.01364						
1	.2560	.0815								
2	.05125									
3	.01026	.00006								

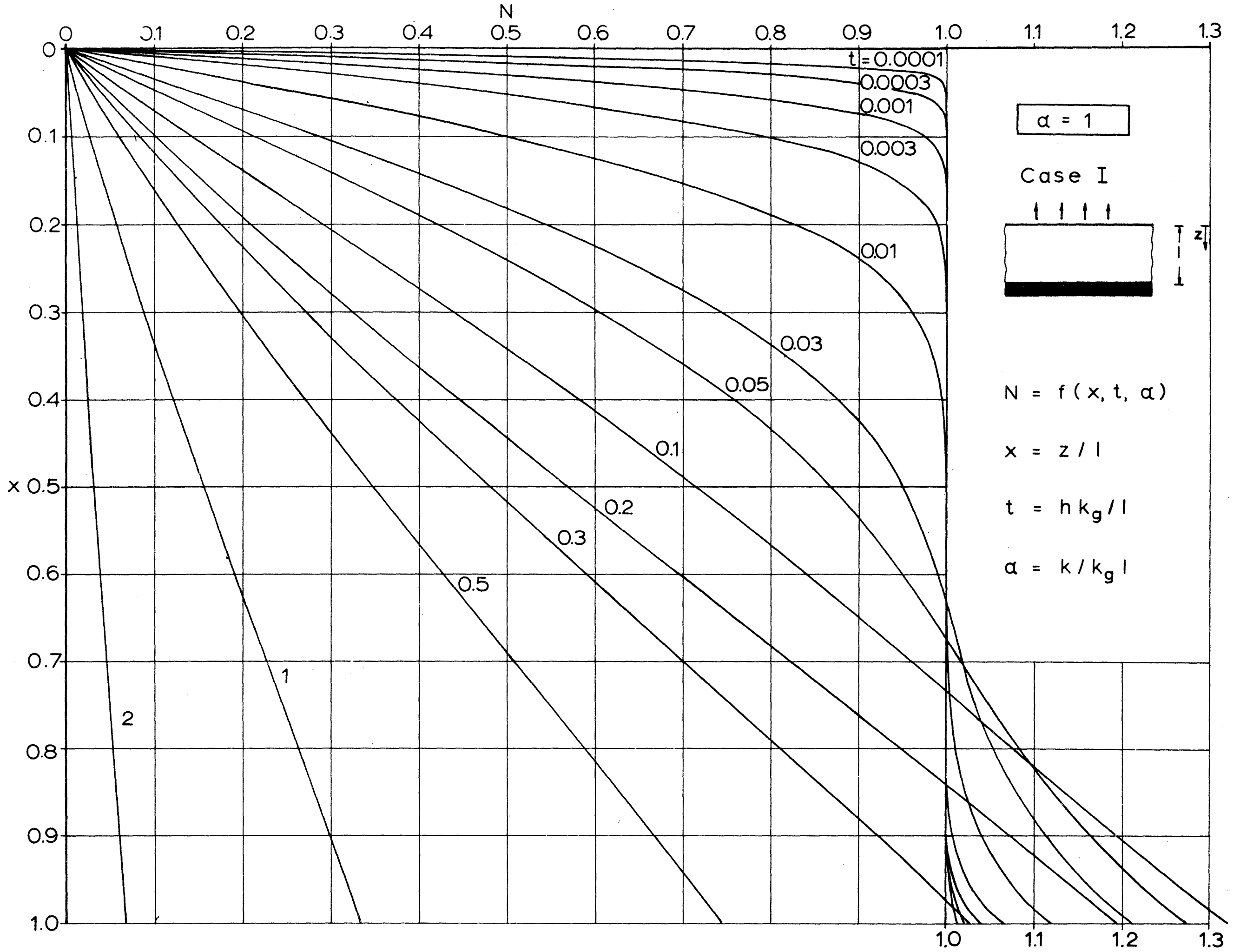
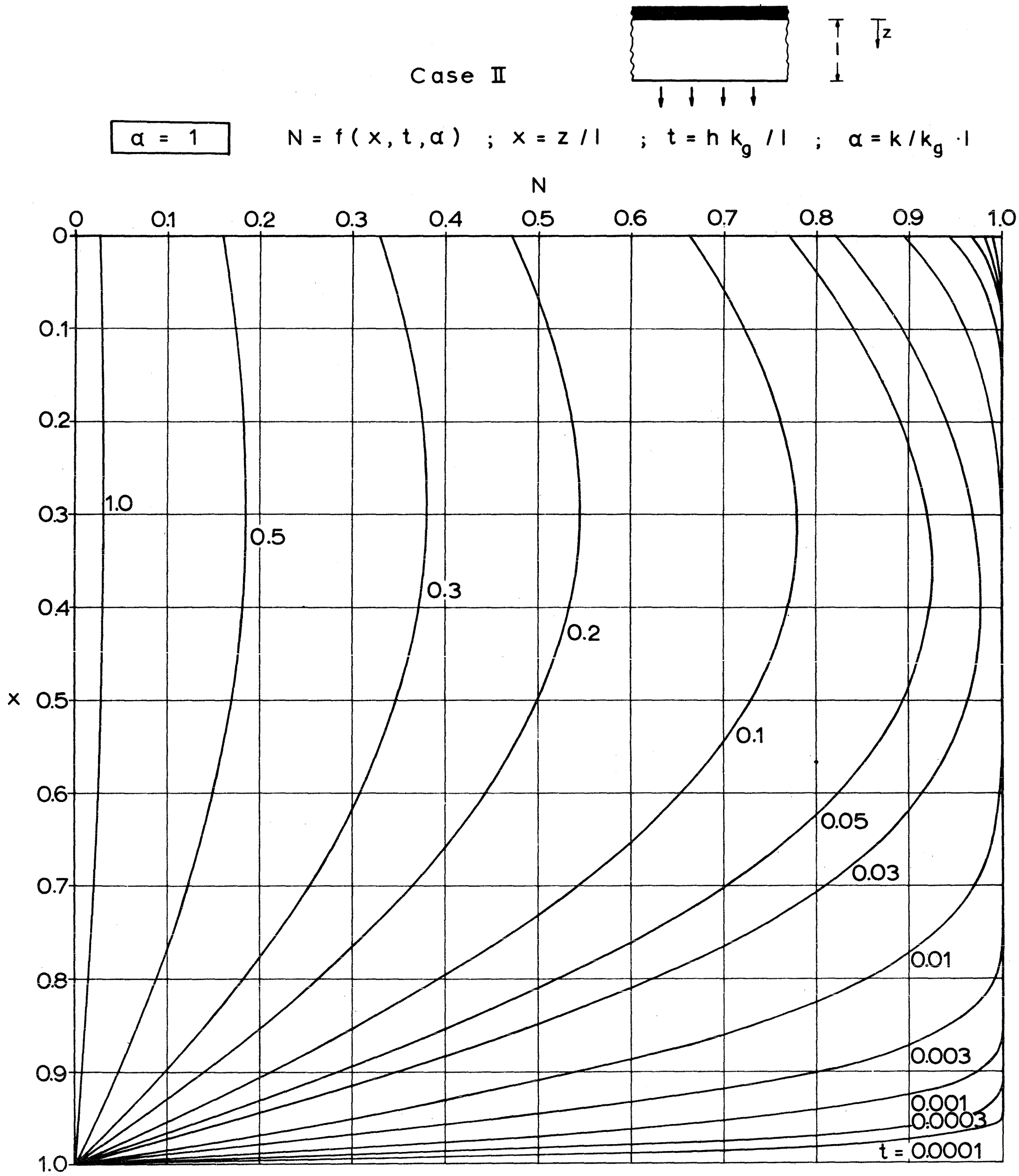

N
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
x
t = 0.0001
0.0003
0.001
0.003
0.01
0.03
0.05
0.1
0.2
0.3
0.5
1
2
α = 1
Case I
z
l
N = f (x, t, α)
x = z / l
t = h k_g / l
α = k / k_g l

Figure 1.



Case II
z
l
α = 1
N = f (x, t, α) ; x = z / l ; t = h k_g / l ; α = k / k_g · l
N
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
x
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
1.0
0.5
0.3
0.2
0.1
0.05
0.03
0.01
0.003
0.001
0.0003
t = 0.0001

Figure 2.

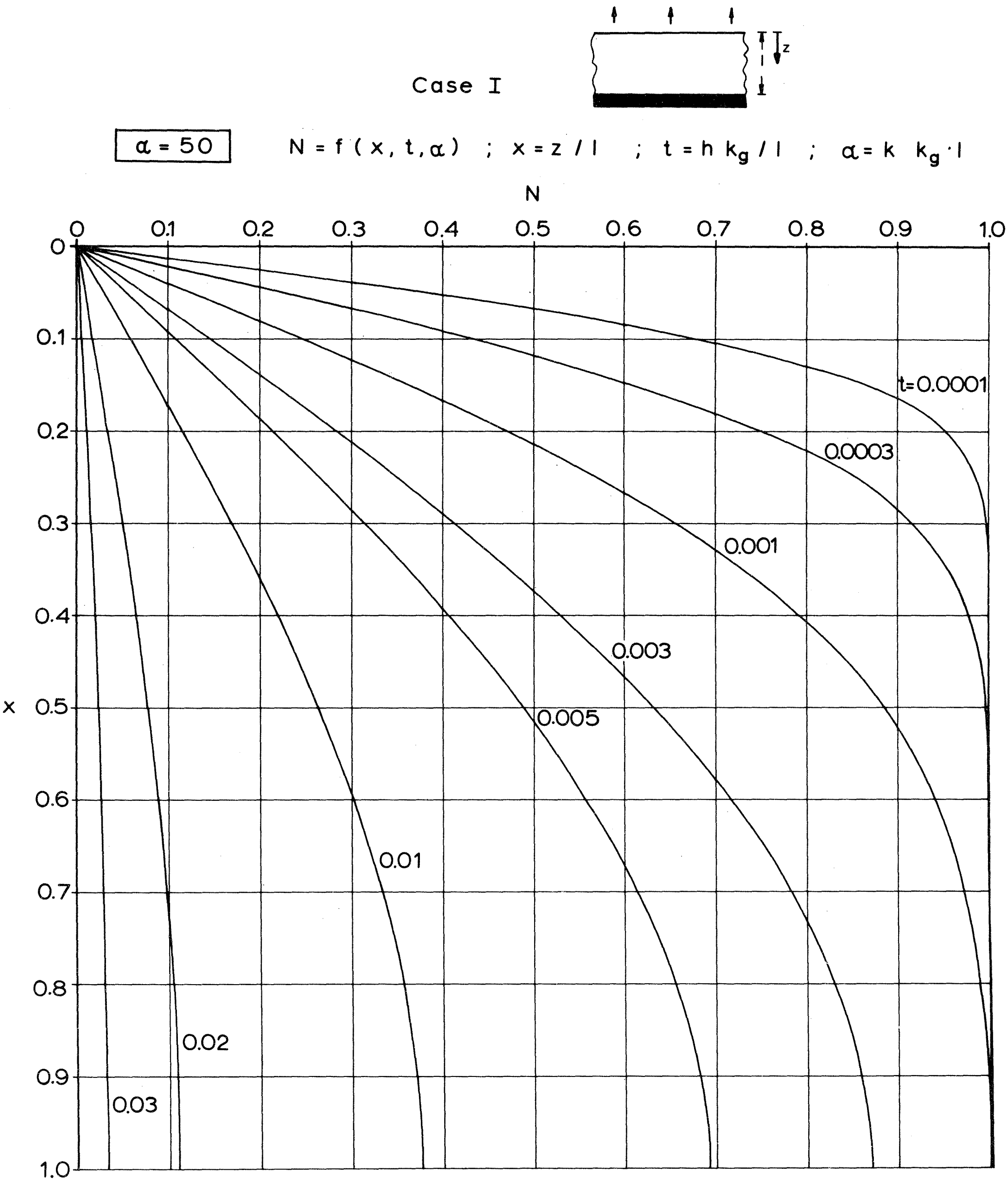

z
l
Case I
α = 50
N = f (x, t, α) ; x = z / l ; t = h k_g / l ; α = k k_g · l
N
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x
t=0.0001
0.0003
0.001
0.003
0.005
0.01
0.02
0.03

Figure 9.

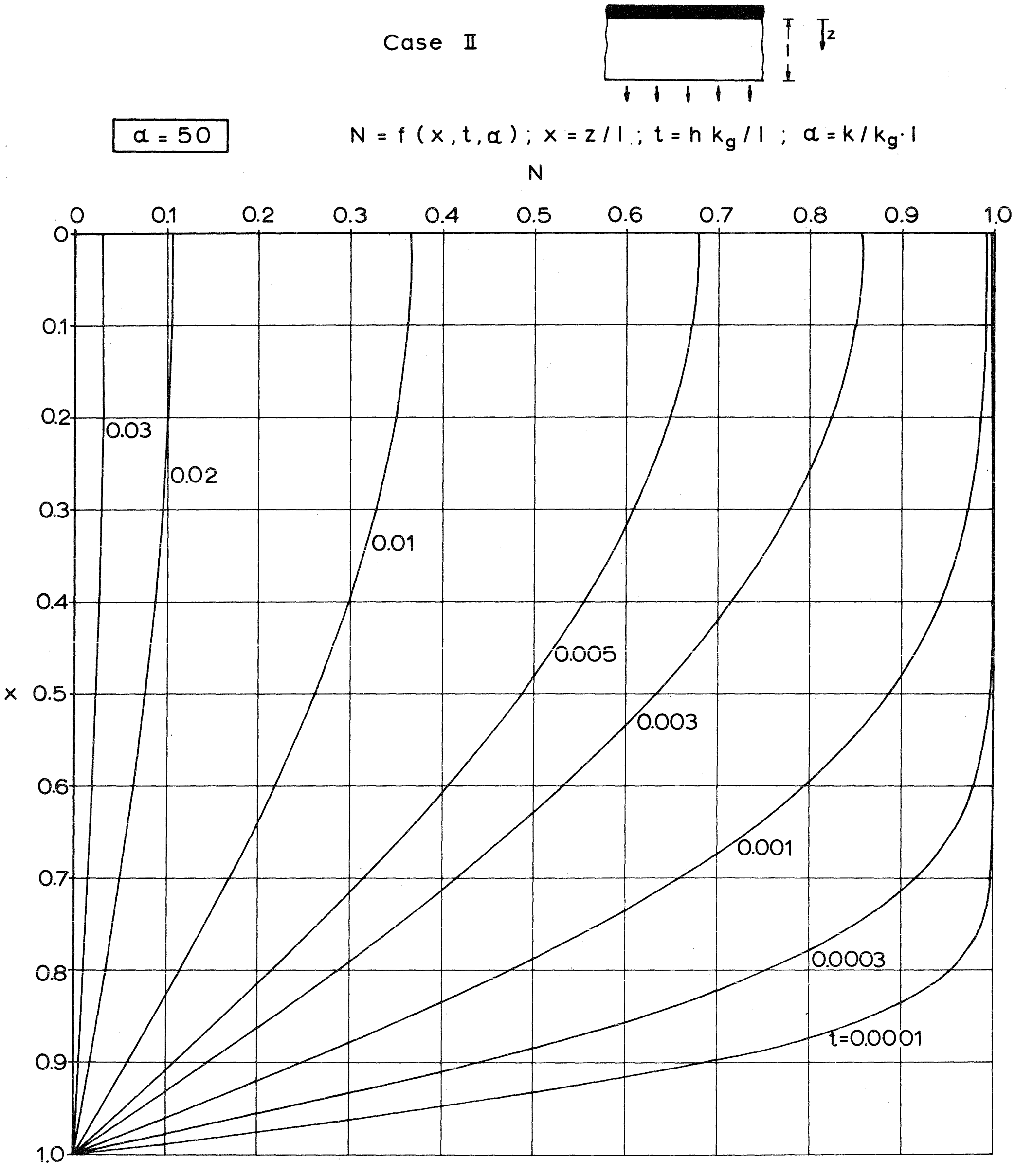
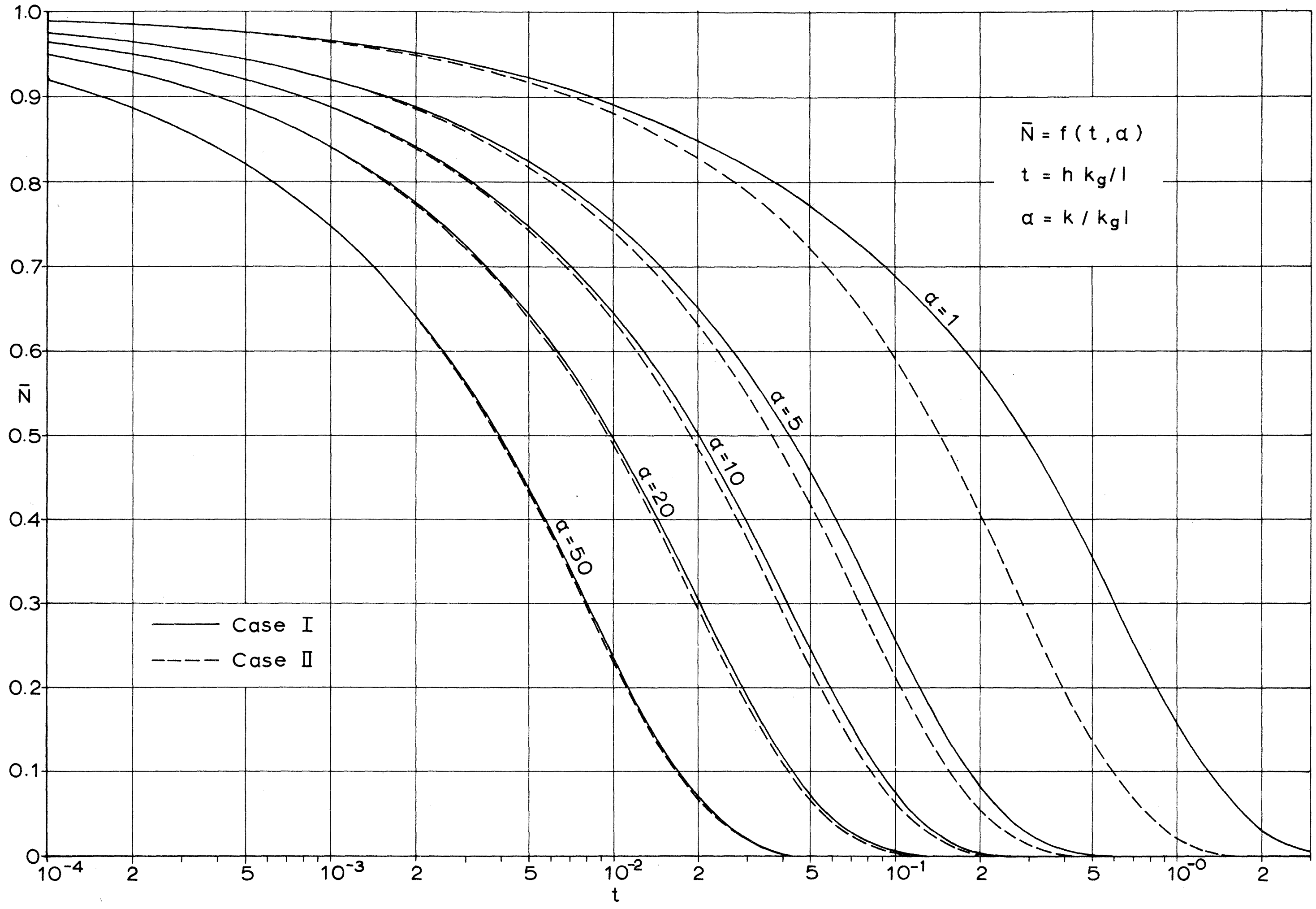
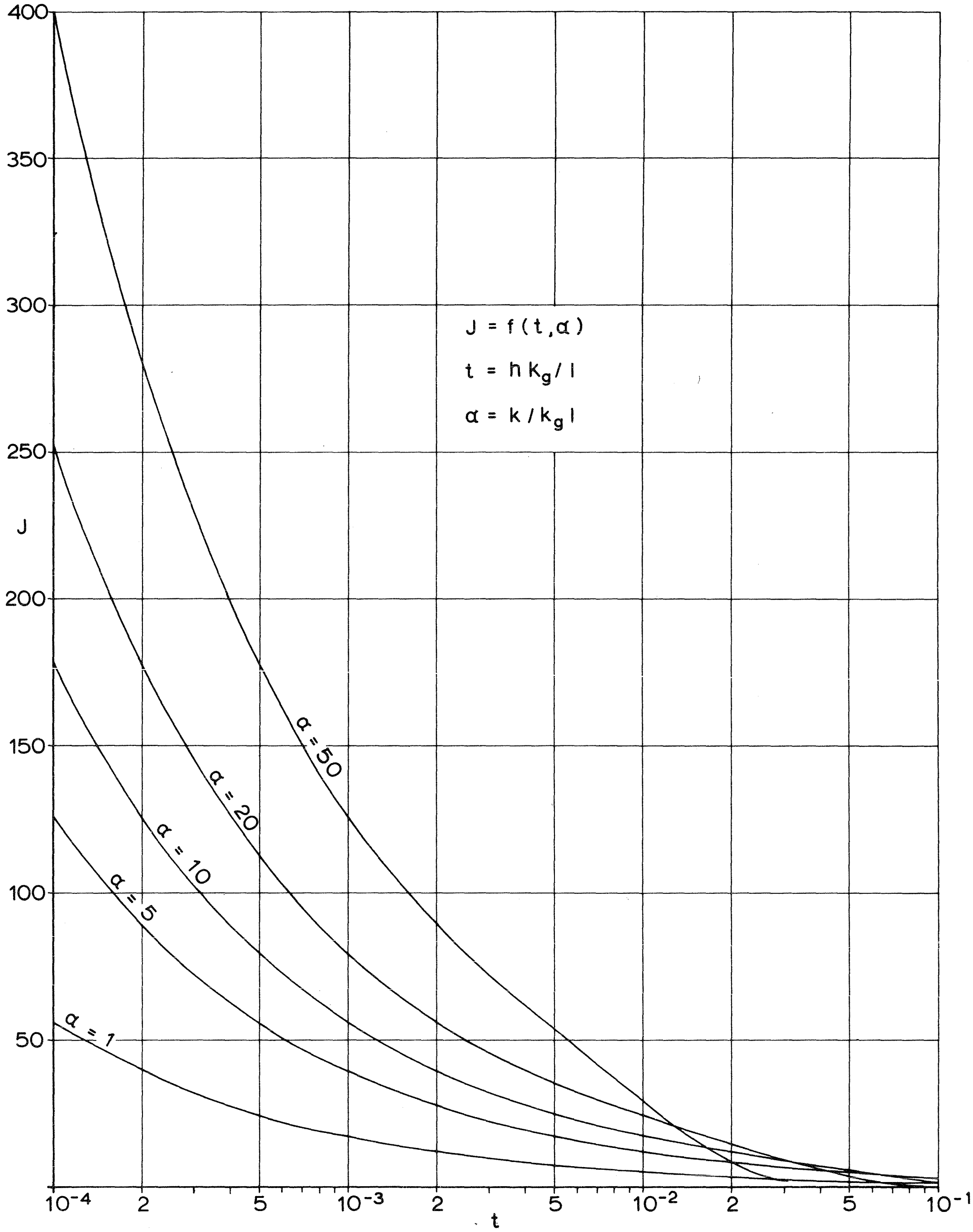

Case II
z
α = 50
N = f (x, t, α) ; x = z / l ; t = h kg / l ; α = k / kg · l
N
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x
0.03
0.02
0.01
0.005
0.003
0.001
0.0003
t=0.0001

Figure 10.



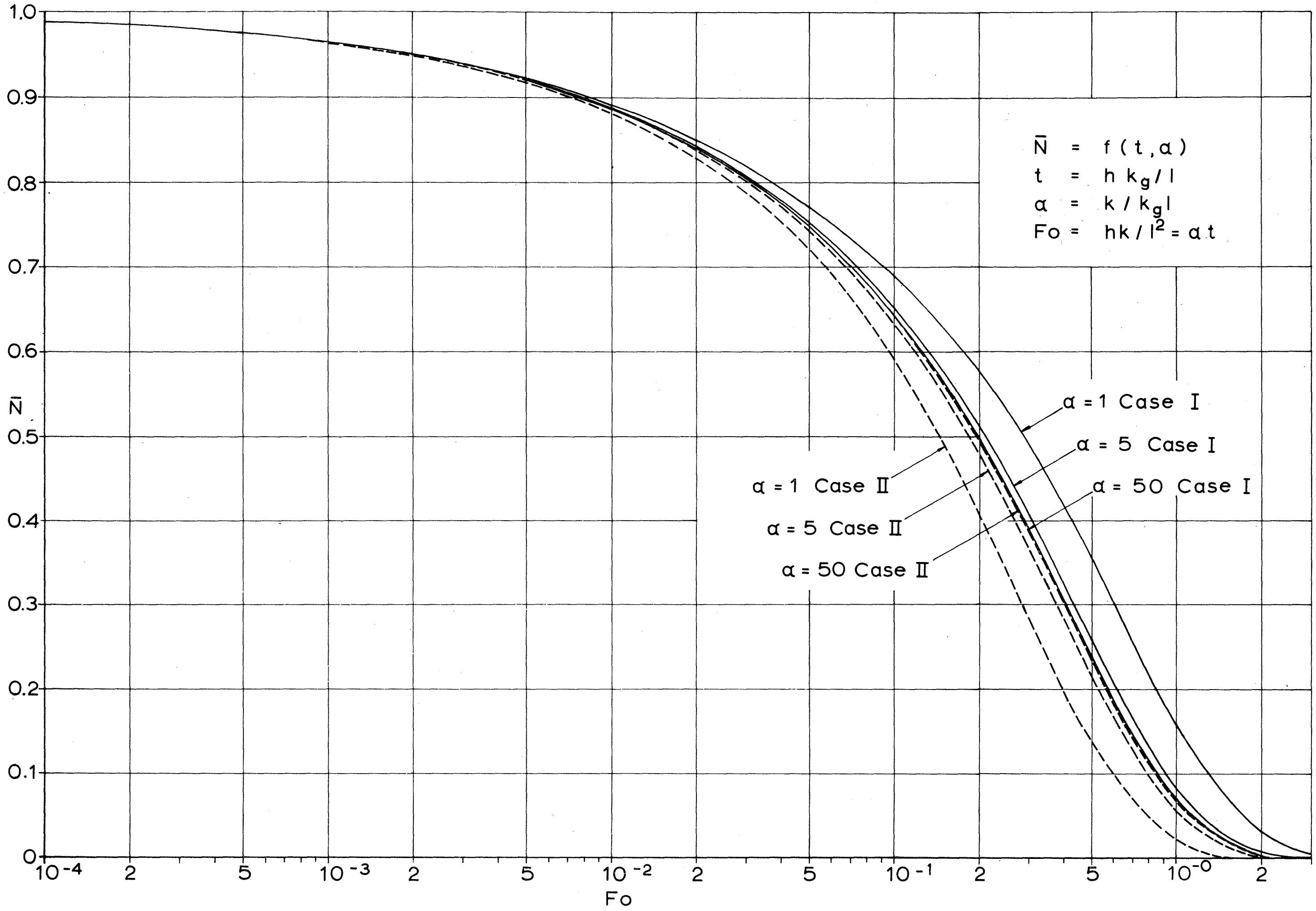
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$\bar{N}$
10^{-4} 2 5 10^{-3} 2 5 10^{-2} 2 5 10^{-1} 2 5 10^{-0} 2
t
$\bar{N} = f(t, \alpha)$
$t = h k_g / l$
$\alpha = k / k_g l$
α = 1
α = 5
α = 10
α = 20
α = 50
Case I
Case II

Figure 11.



400
350
300
250
J
200
150
100
50
$J = f(t, \alpha)$
$t = h\,k_g / l$
$\alpha = k / k_g l$
α = 50
α = 20
α = 10
α = 5
α = 1
10^{-4}
2
5
10^{-3}
2
5
10^{-2}
2
5
10^{-1}
t

Figure 12.



$\bar{N}$ = f (t , α)
t = h k_g / l
α = k / k_g l
Fo = hk / l^2 = α t
α = 1 Case I
α = 5 Case I
α = 50 Case I
α = 1 Case II
α = 5 Case II
α = 50 Case II
$\bar{N}$
Fo
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
10^{-4} 2 5 10^{-3} 2 5 10^{-2} 2 5 10^{-1} 2 5 10^{-0} 2

Figure 13.

4 CONCLUSION

The mathematical formulation and treatment of the diffusive drying of concrete involving the effect of gravitation is essentially based on the assumption that cement paste is a stable material in which, therefore, neither hydration (fixation drying) nor carbonation or ageing occur during the drying process. Furthermore, assumptions have been made that both the moisture conductivity k and the gravitational moisture conductivity k are constants, involving also more assumptions on stability.

In spite of the severe restrictions mentioned above it seems that the mathematical description of the phenomenon, as it appears in the figures presented, agrees perhaps surprisingly well with the presumed physical picture of the effects of gravitation on drying. An attempt has been made to clarify the effect of gravitation and it is hoped that this attempt could be considered as at least a small improvement and stimulation towards interpreting the phenomenon in hand.

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JULKAISUJA - PUBLICATIONS

76. Vuorelainen, O: Käytännöllinen menetelmä suoraan maan pinnalle perustettujen rakennusten maaperään suuntautuvien lämpöhäviöiden laskemiseksi. 48 s. A practical method for calculation of the heat losses into the ground from buildings erected immediately on the ground. 51 p. 1963.

77. Keinonen, Leo: On the sensitivity of water-laid sediments in Finland and factors inducing sensitivity. 131 p. 1963.

78. Korhonen, K-H.: Pienten putkijohtojen kuormituksesta. English summary. Loading of small pipelines. 79 s. 1963.

79. Kantee, L.: Maatilaan kuuluvan peltoalueen taloudellisesta käyttöarvosta tilan kannalta. English summary. On the economic productive value of the arable area belonging to a farm, from the viewpoint of the farm. 397 s. 1963.

80. Liiri, Osmo: Lastulevyn puuraaka-aine. Tutkimuksia suomalaisen puuraaka-aineen eräiden ominaisuuksien merkityksestä valmistettaessa kolmikerroksista laakapuristettua lastulevyä. English summary. Wood raw material of particle board. Investigations on the significance of some properties on Finnish wood raw material in the manufacture of 3-layer flat-pressed particle board. 181 s. 1964.

81. Niskanen, Erkki: On the strength and elasticity characteristics of Finnish structural birch plywood. 29 p. 1963.

82. Ylinen, A.: A comparative study of different types of shear test of wood. 67 p. 1963.

83. Salokangas, Jaakko: Die Bestimmung des Restaustenits in Werkzeugstahl nach dem Wirbelstromverfahren. 16 S. 1963.

84. Siimes, F.: Suomalaisia koesahaustuloksia. Tukkikoon ja -laadun sekä sahausasetteen vaikutus määrälliseen ja laadulliseen sahaustulokseen. Sahatukkien ominaisuudet ja ominaisuuksien keskinäiset suhteet. Painossa.

85. Salmela, Arto: Tonttien hinnat ja asemakaavakustannukset. Zusammenfassung. Grundstückspreise und Erschliessungskosten. 123 s. 1964.

86. Ravaska, J.: Tonttien hintojen suhde rakennuskustannuksiin Helsingissä ja Itä-Espoossa vuosina 1953 ··· 1962. Zusammenfassung. Das Verhältnis von Grundstückspreisen und Baukosten in Helsinki und Ost-Espoo in den Jahren 1953 ··· 1962. 82 s. 1964.

87. Visapää, Asko: Analysis of cement by X-ray fluorescence. 121 p. 1964.

88. Hyyppä, J.M.I.: The influence of the quality of mineral aggregates upon the optimum binder content of asphalt concrete pavements, as determined by Hveem's CKE method. 31 p. 1964.