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$$
m = m_c \frac{1 - \rho_0 / \rho_{ref}}{1 - \rho_0 / \rho_{20}} \qquad \frac{a^2(m_c) + a^2(m_c) + a^2(m_c) + b^2 \frac{1}{(\rho m_c)^2 + \rho_c^2} \left[\frac{1}{2} (\gamma_s) + a^2 (\gamma_s)
$$

Basic formulas for mass calibration

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BASIC FORMULAS FOR MASS CALIBRATION

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Contents:

1. Introduction

This document gives basic formulas for weight and balance calibration. Formulas are expressed both in terms of the true mass and in terms of the conventional mass. The purpose of this document is to clarify the difference of these quantities. More detailed formulas have been given in elsewhere /1-5/.

2. Weight calibration

2.1 Conventional mass

The conventional mass of a body $/1/$ is equal to the mass, m_c of a standard that balances this body at conventionally chosen conditions:

- temperature $t_{20} = 20$ °C
- air density $\rho_0 = 1.2 \text{ kg/m}^3$ (at 20 °C)
- density of the standard $\rho_{ref} = 8000 \text{ kg/m}^3$.

The unit of the conventional mass is the kilogram.

According to this definition the relation between the true mass *m* of the body and its conventional mass m_c is the following:

$$
m = m_c \frac{1 - \rho_0 / \rho_{ref}}{1 - \rho_0 / \rho_{20}}
$$
 (1)

$$
m = m_c - \rho_0 \left(\frac{m_c}{\rho_{ref}} - V(t_{20}) \right)
$$
 (1')

$$
m_c = m \frac{1 - \rho_0 / \rho_{20}}{1 - \rho_0 / \rho_{ref}}
$$
 (2')

$$
m_c = \left(1 - \frac{\rho_0}{\rho_{ref}}\right)^{-1} (m - \rho_0 V(t_{20}))
$$
\n(2')

Where:

 $V(t_{20})$ = volume of the weight at reference temperature t_{20} = 20 °C. $\rho_{20} = m/V(t_{20})$ is the density of the weight at the reference temperature.

The formulas are given both in terms of volume and in terms of density. The equations in (1) and (2) do not contain any approximations.

2.2 True mass difference

Two weights, the test weight m_X and reference weight m_R are compared with an ideal mass comparator. The mass difference between the weights is a sum of the air buoyancy difference and the indication difference (see 3.1):

$$
m_X - m_R = \rho_a (V_X(t) - V_R(t)) + f(I_X - I_R)
$$

$$
f = \frac{(m_s - \rho_{as} V_s)}{\Delta I}
$$
 (3)

Where

 m_X = true mass of test weight m_R = true mass of reference weight ρ_a = density of air during comparison $V_R(t)$ = volume of reference weight at temperature *t* $V_X(t)$ = volume of test weight at temperature *t f* = scale factor I_X = indication of balance with the test weight I_R = indication of balance with the reference weight m_s = mass of sensitivity weight V_s = volume of sensitivity weight $\rho_s = m_s / V_s$ = density of the sensitivity weight $\rho_{\rm as}$ = air density during sensitivity measurement ΔI = indication difference due to sensitivity weight

In formula (3) it is assumed that the air density is constant during the comparison and that the centre of gravity for both weights is equal. Other contributions such as convection forces, electrostatic forces and magnetic forces has been neglected.

The definition of the scale factor f is not unambiguous. Here It is determined by placing the sensitivity weight *ms* on the weighing pan and reading the corresponding indication difference ∆*I*. In principle no adjustment is needed. If the indication difference is adjusted equal to the mass of the sensitivity weight $\Delta I = m_s$ then $f = 1 - \rho_{as} / \rho_{s}$.

2.3 Conventional mass difference:

The conventional mass difference of the two weights derived from (3) is the following:

$$
m_{X,c} - m_{R,c} = \frac{(\rho_a - \rho_0)(V_X(t) - V_R(t)) + \rho_0(V_X(t_0)\gamma_x - V_R(t_0)\gamma_R)(t - t_0)}{1 - \frac{\rho_0}{\rho_{ref}}} + f_C(I_X - I_R)
$$

$$
f_C = \frac{(m_s - \rho_{as}V_s)}{\left(1 - \frac{\rho_0}{\rho_{ref}}\right)\Delta I} = (m_{s,c} - \frac{(\rho_{as} - \rho_0)V_s}{\left(1 - \frac{\rho_0}{\rho_{ref}}\right)})\frac{1}{\Delta I}
$$
(4)

Where

 $m_{X,c}$ = conventional mass of test weight $m_{R,c}$ = conventional mass of reference weight ρ_a = density of air during comparison γ_R = volume expansion coefficient of the reference weight γ_X = volume expansion coefficient of the test weight f_C = scale factor $m_{\rm sc}$ = conventional mass of sensitivity weight

The scale factor f_c is the coefficient between conventional mass and the corresponding indication. Its value is different from the scale factor for the true mass. If the indication difference ΔI is adjusted equal to $m_{s,c}$ then $f_c \approx 1 - (\rho_{as} - \rho_0)/\rho_s$.

2.4 True mass and its uncertainty

From (3) the following equation for m_X can be derived. If the mass of the reference weight has been determined in a comparison with a higher order mass standard m_{ST} then an analogous formula for m_R can be derived:

$$
m_R = m_{ST} + \rho_{ar}(V_R - V_{ST}) + f_r(I_R - I_S)
$$

\n
$$
m_X = m_R + \rho_a(V_X - V_R) + f(I_X - I_R)
$$
\n(5)

Where:

 m_{ST} = mass of the higher order mass standard V_{ST} = volume of the higher order mass standard ρ_{ar} = density of air when m_R was calibrated

The covariance between the true mass and the volume of the reference weight $u(m_R, V_R)$ has the following form:

$$
u(m_R, V_R) = \frac{\partial m_R}{\partial V_R} \frac{\partial V_R}{\partial V_R} u^2(V_R) = \rho_{ar} u^2(V_R)
$$
\n⁽⁶⁾

From (5) the square of the standard uncertainty $u(m_X)$ of m_X is:

9

$$
u^{2}(m_{X}) = u^{2}(m_{R}) + u^{2}(\rho_{a})(V_{X} - V_{R})^{2} + \rho_{a}^{2}[u^{2}(V_{X}) + u^{2}(V_{R})] + u^{2}(f)(I_{X} - I_{R})^{2}
$$

+ $f^{2}u^{2}(I_{X} - I_{R}) + 2\frac{\partial m_{X}}{\partial m_{R}}\frac{\partial m_{X}}{\partial V_{R}}u(m_{R}, V_{R})$

$$
u^{2}(m_{X}) = u^{2}(m_{R}) + u^{2}(\rho_{a})(V_{X} - V_{R})^{2} + \rho_{a}^{2}[u^{2}(V_{X}) + u^{2}(V_{R})] + u^{2}(f)(I_{X} - I_{R})^{2}
$$

+ $f^{2}u^{2}(I_{X} - I_{R}) - 2\rho_{a}\rho_{ar}u^{2}(V_{R})$ (7)

Where $u(x_i)$ is the standard uncertainty of the quantity x_i .

$$
u(x_i, x_j) = \sum_{k} \frac{\partial X_i}{\partial x_k} \frac{\partial X_j}{\partial x_k} u^2(x_k)
$$
 is the covariance between x_i and x_j .

The correlation coefficient of x_i and x_j is $r(x_i, x_j) = u(x_i, x_j)/(u(x_i)u(x_j))$. The correlation coefficient between m_X and V_X is

$$
r(m_X, V_X) = \frac{u(m_X, V_X)}{u(m_X)u(V_X)} = \rho_a \frac{u(V_X)}{u(m_X)}
$$
(8)

It is assumed that there is no other correlation.

2.5 The conventional mass and its uncertainty

In the following the same formulas as in the previous section are given in terms of the conventional mass. The conventional mass of the reference weight $m_{R,C}$ has been determined with a higher order mass standard $(m_{s,C}, V_s)$ at air density ρ_a . The conventional masses can be obtained from the following equations derived from (4):

$$
m_{R,C} \approx m_{ST,C} + \frac{(\rho_{ar} - \rho_0)(V_R - V_S)}{1 - \frac{\rho_0}{\rho_{ref}}} + f_{Cr}(I_R - I_S)
$$

$$
m_{X,C} \approx m_{R,C} + \frac{(\rho_a - \rho_0)(V_X - V_R)}{1 - \frac{\rho_0}{\rho_{ref}}} + f_C(I_X - I_R)
$$
 (9)

The conventional mass $m_{R,C}$ and the volume V_R are correlated their covariance $u(m_{R,C},V_R)$ is:

$$
u(m_{R,C},V_R) = \frac{\partial m_{R,C}}{\partial V_R} \frac{\partial V_R}{\partial V_R} u^2(V_R) = \frac{(\rho_{ar} - \rho_0)}{1 - \frac{\rho_0}{\rho_{ref}}} u^2(V_R)
$$
(10)

The standard uncertainty $u(m_{X,c})$ can be obtained from the formula:

$$
u^{2}(m_{X,c}) = u^{2}(m_{R,c}) + \frac{u^{2}(\rho_{a})}{(1 - \frac{\rho_{0}}{\rho_{ref}})^{2}}(V_{X} - V_{R})^{2} + \frac{(\rho_{a} - \rho_{0})^{2}}{(1 - \frac{\rho_{0}}{\rho_{ref}})^{2}}\left[u^{2}(V_{X}) + u^{2}(V_{R})\right] + u^{2}(f_{C})(I_{X} - I_{R})^{2} + f_{C}^{2}u^{2}(I_{X} - I_{R}) + 2\frac{\partial m_{X,C}}{\partial m_{X,c}}\frac{\partial m_{X,c}}{\partial V_{R}}u(m_{R,c}, V_{R}) + u^{2}(\rho_{a})(u^{2}(V_{X}) + u^{2}(V_{R})) \tag{11}
$$

Both the covariance term (second term on last line) and a second order term (last term) have been included. The second order term is significant only if $V_x \approx V_R$ and ρ_a $\approx \rho_0$.

If we assume that the constant $1-\rho_0/\rho_{ref} \approx 1$ then

$$
u^{2}(m_{X,C}) \approx u^{2}(m_{R,C}) + u^{2}(a)(V_{X} - V_{R})^{2} + (\rho_{a} - \rho_{0})^{2} [u^{2}(V_{X}) + u^{2}(V_{R})] + u^{2}(f_{C})(I_{X} - I_{R})^{2}
$$

+ $f_{C}^{2}u^{2}(I_{X} - I_{R})^{2} - 2(\rho_{a} - \rho_{0})(\rho_{ar} - \rho_{0})u^{2}(V_{R}) + u^{2}(a)(u^{2}(V_{X}) + u^{2}(V_{R}))$

Also $m_{X,C}$ and V_X are correlated the correlation coefficient is

$$
r(m_{X,C}, V_X) = \frac{u(m_{X,C}, V_X)}{u(m_{X,C})u(V_X)} = \frac{(\rho_a - \rho_0)}{1 - \frac{\rho_0}{\rho_{ref}}} \frac{u(V_X)}{u(m_{X,C})}
$$
(13)

2.6 Mass expressed in terms of density

From (5) the true mass m_x expressed in terms of density is

$$
m_x = m_R \frac{1 - \rho_a / \rho_R}{1 - \rho_a / \rho_X} + f(I_X - I_R) \frac{1}{1 - \rho_a / \rho_X}
$$
 (5')

From (9) The corresponding formula for the conventional mass is

$$
m_{x,c} = m_{R,c} \frac{1 - \rho_a / \rho_R}{1 - \rho_a / \rho_X} \frac{1 - \rho_0 / \rho_X}{1 - \rho_0 / \rho_R} + f_c (I_X - I_R) \frac{1 - \rho_0 / \rho_X}{1 - \rho_a / \rho_X}
$$
(9')

The covariance between mass and density is approximately:

$$
u(m_X, \rho_X) = \frac{\partial m_X}{\partial \rho_X} \frac{\partial \rho_X}{\partial \rho_X} u^2(\rho_X) = -m_X \frac{\rho_a}{\rho_X(\rho_X - \rho_a)} u^2(\rho_X)
$$
(8')

The covariance between conventional mass and density is approximately:

$$
u(m_{X,c}, \rho_X) = \frac{\partial m_{X,c}}{\partial \rho_X} \frac{\partial \rho_X}{\partial \rho_X} u^2(\rho_X) = -m_{X,c} \frac{(\rho_a - \rho_0)}{(\rho_X - \rho_0)(\rho_X - \rho_a)} u^2(\rho_X)
$$
(10')

3. Mass comparators

3.1 Model for a mass comparator

Gravitational force F due to a weight m_t is:

$$
\frac{F}{g} = m_t \left(1 - \frac{\rho_a}{\rho_t}\right) \tag{14}
$$

 m_t = mass of weighted object ρ_t = density of weighted object ρ_a = air density during weighing F = vertical force to the weighing pan *g* = acceleration of free fall

The indication *I* of an ideal balance when loaded with mass m_t is directly proportional to the gravitational force:

$$
I(m_t) = \alpha \cdot \frac{F}{g} = \alpha \cdot m_t (1 - \rho_a / \rho_t)
$$
\n(15)

Where:

I = indication of the balance α = calibration constant

The calibration constant α is determined by weighing the adjustment weight m_{cs} and by setting the display equal to *mcs*.

$$
m_{cs} = \alpha \cdot m_s (1 - \rho_{as} / \rho_s) = \alpha \cdot m_{cs} \frac{1 - \rho_0 / \rho_{ref}}{1 - \rho_0 / \rho_s} (1 - \rho_{as} / \rho_s)
$$
(16)

$$
\alpha^{-1} = \left[\frac{1 - \rho_0 / \rho_{ref}}{1 - \rho_0 / \rho_s} (1 - \rho_{as} / \rho_s) \right]
$$
(17)

Where:

 ρ_{as} = air density during adjustment *ms =* mass of the adjustment weight m_{cs} = conventional mass of adjustment weight ρ_s = density of adjustment weight

By substituting α to (15) and solving m_t we have

$$
m_{t} = \frac{1 - \rho_{as} / \rho_{s}}{1 - \rho_{a} / \rho_{t}} \cdot \frac{1 - \rho_{0} / \rho_{ref}}{1 - \rho_{0} / \rho_{s}} I(m_{t})
$$
(18)

$$
m_t \approx \left[1 + \left(\frac{1}{\rho_t} - \frac{1}{\rho_s}\right)\rho_a - \frac{\rho_{as} - \rho_a}{\rho_s}\right] \cdot \left[\frac{1 - \rho_0/\rho_{ref}}{1 - \rho_0/\rho_s}\right] \cdot I(m_t)
$$
(18')

In (18) m_t is can be expressed in terms of conventional mass m_{ct} .

$$
m_{t} = m_{ct} \frac{(1 - \rho_{0} / \rho_{ref})}{1 - \rho_{0} / \rho_{t}}
$$
(19)

When m_{ct} is solved we get

$$
m_{tc} = \frac{1 - \rho_{as} / \rho_s}{1 - \rho_a / \rho_t} \cdot \frac{1 - \rho_0 / \rho_t}{1 - \rho_0 / \rho_s} \cdot I(m_t)
$$
(20)

$$
m_{tc} \approx \left[1 + \left(\frac{1}{\rho_t} - \frac{1}{\rho_s}\right) \cdot (\rho_a - \rho_0) - \frac{\rho_{as} - \rho_a}{\rho_s}\right] \cdot I(m_t)
$$
 (20')

Corrections to the approximations (18') and (20') are of the order of $(\rho_a/\rho_f)^2$ *I*(m_t). In practice they can be neglected. If the balance is adjusted before the measurement then $\rho_{as} \approx \rho_a$ and the last terms on the right hand side of (18') and (20') can be neglected.

When calibrating the balance the following models are possible:

a) The error of indication of the balance without air buoyancy correction *E* is determined:

$$
E = I(m_{tc}) - m_{tc} + \delta I(m_{tc}) + \delta m_{tc}
$$
\n(21)

where $\delta I(m_t)$ is a correction due to nonidealities of the balance (eccentricity, magnetic effects etc.) at load m_t and δm_{tc} is zero correction due to air buoyancy. It is included to be taken into account in uncertainty calculations.

b) The error of air buoyancy corrected indication is determined:

$$
E = I(m_t) \cdot \frac{1 - \rho_{as} / \rho_s}{1 - \rho_a / \rho_t} \frac{1 - \rho_{a0} / \rho_{ref}}{1 - \rho_{a0} / \rho_s} - m_t + \delta I(m_t)
$$
(22)

or

$$
E = I(m_{ic}) \cdot \frac{1 - \rho_{as} / \rho_s}{1 - \rho_a / \rho_t} \cdot \frac{1 - \rho_{a0} / \rho_t}{1 - \rho_{a0} / \rho_s} - m_{ic} + \delta I(m_{ic})
$$
 (23)

In practice the formula (23) is preferred because the density and the (true) mass of the weight are strongly correlated.

3.2 Uncertainty of the error of indication

For uncertainty calculations In (22) and (23) it can be assumed that the density of the sensitivity weight is equal to the reference density: $\rho_s = \rho_{ref}$. In addition the following approximations will be made:

$$
E \approx I(m_t) + I(m_t)(\rho_a/\rho_t - \rho_{as}/\rho_{ref}) - m_t + \delta I(m_t)
$$
\n(24)

and

$$
E \approx I(m_{ic}) + I(m_{ic})((\rho_0 - \rho_{as})/\rho_{ref} + (\rho_a - \rho_0)/\rho_t) - m_{ic} + \delta I(m_{ic})
$$
 (25)

The uncertainty of *E* from (24) is calculated assuming that ρ_{as} is constants and $I(m_t) \approx$ $m_t \approx m_{t,c}$. The air density at the time when the weights were calibrated is ρ_{at} .

$$
u^{2}(E) = u^{2}(I(m_{t})) + I^{2}(m_{t}) \cdot \left[\left(\frac{u^{2}(\rho_{a})}{\rho_{t}^{2}} + \frac{\rho_{a}^{2}}{\rho_{t}^{4}} \cdot u^{2}(\rho_{t}) \right) + u^{2}(m_{t}) + u^{2}(\delta I) - 2I(m_{t})^{2} \frac{\rho_{a}\rho_{at}}{\rho_{t}^{4}} u^{2}(\rho_{t}) \right]
$$
(26)

Here the covariance term with (8')

$$
2\frac{\partial E}{\partial m_t} \frac{\partial E}{\partial \rho_t} u(m_t, \rho_t) \approx 2 \cdot (-1) \cdot \left[-I(m_t) \frac{\rho_a}{\rho_t^2} \right] \cdot \left[-\frac{\rho_{at}}{\rho_t^2} m_t u^2(\rho_t) \right]
$$
(27)

has been included.

For conventional mass from (25):

$$
u^{2}(E) = u^{2}(I(m_{ic})) + I^{2}(m_{ic}) \cdot \left[\frac{u^{2}(\rho_{a})}{\rho_{i}^{2}} + \frac{(\rho_{a} - \rho_{0})^{2}}{\rho_{i}^{4}} \cdot u^{2}(\rho_{i}) \right] + u^{2}(m_{ic})
$$

+
$$
u^{2}(\delta I) - 2I(m_{i})^{2} \frac{(\rho_{a} - \rho_{0})(\rho_{a1} - \rho_{0})}{\rho_{i}^{4}} u^{2}(\rho_{i})
$$
(28)

Also here the covariance term with (10'):

$$
2\frac{\partial E}{\partial m_{ic}}\frac{\partial E}{\partial \rho_{t}}u(m_{t},\rho_{t}) = 2\cdot(-1)\cdot\left[-I(m_{t})\frac{(\rho_{a}-\rho_{0})}{\rho_{t}^{2}}\right]\cdot\left[-\frac{(\rho_{a1}-\rho_{0})}{\rho_{t}^{2}}m_{t}u^{2}(\rho_{t})\right] (29)
$$

has been included.

Formulas (26) and (28) give practically identical values for the uncertainty of the error *E* .

3.3 Modified scale factor

Usually the mass of the weighed object is of primary interest. In addition to the mass the balance indication also depends on air density and on the density of the weighed object. In principle the density dependence can be corrected. Because a balance is not an ideal instrument some additional corrections are needed. One possibility is to divide the scale factor into two components. From (22) if we assume that $\rho_s = \rho_{ref}$ we have

$$
m_t = I \cdot \frac{1 - \rho_{as} / \rho_{ref}}{1 - \rho_a / \rho_t} - E(I) = f(\rho_a, \rho_t) f_x(I) \cdot I
$$
\n(30)

where the density dependent scale factor $f(\rho_a, \rho_t)$ is

$$
f(\rho_a, \rho_t) = \frac{1 - \rho_{as} / \rho_{ref}}{1 - \rho_a / \rho_t}
$$
\n(31)

and the instrumental scale factor f_x is

$$
f_x(I) \approx 1 - E(I)/I \tag{32}
$$

The instrumental scale factor will be determined in calibration. It contains e.g. nonlinearity of the indication, eccentric loading, temperature dependence of indication. Its value is not however constant with time.

It is important to note that a balance (or a mass comparator) measures force to the weighing pan. It can not directly take into account changes in air density, in the density of the measured object or changes in gravitational acceleration. Changes in air density or in gravitation can be taken into account by adjusting the reading of the balance with an external or internal sensitivity weight. In some balances this is done automatically.

The balance indication is closer to the conventional mass than to the true mass. In many cases the indication can directly used as the conventional mass. This is normally not valid for true mass.

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/4/ M. Kochsiek, M. Gläser (eds.), *Comprehensive Mass Metrology*, Wiley-VCH, 2000, ISBN 3-527-29614-X

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