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Abstract

Modelling of thermal behaviour of buildings needs effective tools. This is particularly true when conduction of heat through slabs and/or walls is computed. The paper proposes a novel approach for such applications. The method is based on differential equation of heat conduction which is further modified to a differential-difference equation with continuous space variable and discrete time variable. The approach differs from conventional differential-difference solutions. In this paper, one-dimensional problems are examined in semi-infinite, one- and multi-layer environment.

Characteristic of the method is that solutions are presented using past values of boundary functions. In addition, transfer functions which determine the response at each time instant are calculated recursively. Because the differential-difference solution is partly numerical, better accuracy is achieved by using analytical methods, such as the pulse transfer method. However, in a multi-layer environment the latter turns out to be more complicated, since several transcendental equations must be solved, contrary to the proposed method.

The differential-difference method is compared with numerical solutions choosing the explicit method as a representative of them. The results show that in most cases better accuracy is achieved with the differential-difference method when time steps of both methods are equal. In addition, the proposed method needs no nodal points inside the slab during computation. Thus, time steps need not be adjusted according to thin layers of the wall, which makes the method feasible in multi-layer environment. The differential-difference approach is inherently stable, which is not true for all numerical methods. The method is suggested to be applied in dynamic thermal models of buildings in which time step is less than one hour.

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Nomenclature

A	area
A_n	constant, matrix element
$A(z)$	matrix element, power series
B_n	constant, matrix element
$B(z)$	matrix element, power series
Bi	Biot number
C	bulk thermal capacitance ($=\rho Ac$)
C_i	lumped thermal capacitance of node i
$\dot{C}_e$	capacitive flow of exhaust air
$\dot{C}_f$	capacitive flow of filtration air
C_n	matrix element
C_R	thermal capacitance of zone air
$C(z)$	matrix element, power series
D_n	matrix element
$D(z)$	matrix element, power series
$(\pi/2x)^{1/2} I_{n+1/2}(x)$	Modified Spherical Bessel function of the first kind
$J(x,t-\eta)$	integral function, defined in text
$(\pi/2x)^{1/2} K_{n+1/2}(x)$	Modified Spherical Bessel function of the third kind
L	thickness of slab
$L\{x\}$	Laplace-transform of x
M	modulus of the explicit numerical method
$O(h)$	truncation error magnitude order of h
P	steady state heat flow rate
$P_n(t)$	heat flow rate at boundary of solid
Q	heat flux
R	bulk thermal resistance ($=1/kA$)
R_i	lumped thermal resistance of node i
R_n	matrix element
RC	time constant of slab
$U_n(t)$	boundary surface temperature
$U_o(t)$	outdoor temperature
$U_R(t)$	room temperature
V_n	matrix element
W_n	matrix element
$a_n(x)$	coefficient function
$a_{m,n}(x)$	coefficient function
$b_n(x)$	coefficient function
$b_{m,n}(x)$	coefficient function
c	specific heat
$c_{m,n}(x)$	coefficient function of particular solution

$g(x,t)$	forcing function
h	lumping constant
h_s	convective coefficient of heat transfer
h_{si}	convective coefficient of heat transfer on surface i
k	conductivity
p	variable of the Laplace transform in x-space
q	constant
r	variable
s	variable of the Laplace transform in t-space
t	time variable
$\hat{u}_i^n$	approximate temperature function based on finite difference solution, i and n refer to nodal points of x and t
$u(x,t)$	temperature function of x and t
v	variable
$v_{m,n}$	coefficient of power series
$w_{m,n}$	coefficient of power series
x	space variable
z	back shift operator
η	displacement constant
κ	diffusivity
μ	constant
ρ	density
φ	convective load

1 Introduction

1.1 Background

Conduction of heat through slabs and walls, originally a topic of physics, has become an area of interest in building services. The reason behind such a trend is the fast development of computer technology which allows large scale problems to be solved with small computers. Today, architects, engineers, and technicians control total buildings with software. A variety of systems and programs is available for design, maintenance, and control of buildings. Many of them have been implemented to deal with dynamical thermal behaviour of buildings and their structures.

Increasing capacity of computers, microprocessors and available memory capacity combined with low price creates opportunities for sophisticated technical systems, also suited to building services. Control systems are developing to building optimization systems (Kelly 1988). Besides ordinary control and energy management they can optimize energy consumption, locate possible faults or detect energy leakages (Pakanen 1992). Prediction of energy consumption and control decisions are based on knowledge of dynamical thermal response to physical quantities inside and outside of the building.

Thermal modelling of buildings has been intensive since the advent of computers. The first attempts were analogical models constructed of real resistors and capacitors. Voltages and currents of the electrical network corresponded temperatures and heat flow rates (Day 1982). The advantage was higher computational speed compared with the models running in computers of early stage. Other approaches such as analytical, finite difference and lumped parameter systems have also their advantages and supporters (Cowan 1976, Mehta 1980). Thermal modelling and simulation concentrated first on building envelope. Later interest focused on internal gains, multi-zone environment, solar gains and shadowing, air infiltration and air movement between rooms (Walton 1983a), thermal comfort algorithms, HVAC-systems and control, as well as nonlinear dynamics and control (Roberts & Oak 1991). As a result, a lot of computer programs for simulation and emulation (May & Park 1985) have been produced which help the design, maintenance, control, and research of buildings.

Conduction of heat through slabs and walls is only one of the physical phenomena necessary to formulate in order to carry out a thermal simulation of a building or zone. Moreover, conduction is only an approximation of the total mass and heat transfer through a slab and most methods apply only to homogeneous, isotropic solids. Although these restrictions influence the accuracy, an equally important question is the efficiency of the method to compute the heat flow. A larger building may contain hundreds of slabs and the software package running the simulation may be massive. An inefficient computing algorithm may ruin the original idea of the system.

1.2 Earlier results

Conduction of heat through a slab is a classical problem mainly solved by two types of methods. The first category includes analytical solutions. If the heat flow through a slab or wall is to be determined with good accuracy, one would apply the pulse response method (Stephenson & Mitalas 1971, Butler 1984, Walton 1984) or other analytical methods such as those presented by Title (1965), Bulavin & Kaschcheev (1965) and Mikhailov et al. (1982). If the number of layers do not exceed two or three and some of the layers are infinite in thickness, one may apply the integral transform techniques (Lykov & Mikhailov 1965). A common feature of these methods is the eigenvalue problem. The complete solution can be expressed with the aid of eigenvalues. In order to determine a necessary amount of them a numerical algorithm is needed which finds roots of the transcendental equations.

Substitution of finite-difference approximation in the diffusion equation has evolved a large number of methods for boundary value problems of heat conduction. Such numerical methods have been extensively applied also to multi-layer slabs. If one takes care of stability and accepts errors involved, heat flow computations with these simple numerical algorithms become successful. A disadvantage is the need of nodal points inside the solid which must be included in the algorithm. When good accuracy is a requirement, inner nodal points increase execution time and restrict the application areas of the algorithm.

1.3 The proposed method

The paper presents a method for boundary value problems of heat conduction that is partly analytical and partly numerical. This is accomplished by changing the differential equation of heat conduction into a differential-difference equation where the space variable is analytical and the time variable discrete. The approach leads to a linear second order differential equation with constant coefficients. Solutions are obtained recursively. Thus, past values of initial and boundary functions can be utilized effectively. No eigenvalues need to be solved and no inner nodal points used. The method is suitable for approximate solutions of diffusion type partial differential equations.

2 Solution of the differential equation of heat conduction

2.1 Deffinition of the initial and boundary value problem

Conduction of heat through a slab is described by a second order partial differential equation of parabolic type. The following discussion is restricted to one-dimensional, linear heat flow through a homogeneous, isotropic solid. If the thermal diffusivity is replaced by a bulk thermal resistance *R* and capacitance *C*, the non-homogeneous differential equation of heat conduction is written as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = RC\frac{\partial u(x,t)}{\partial t} + g(x,t)\,, \qquad (1)$$

where *g(x,t)* represents a forcing function. A solution of Equation (1) in a finite solid in the region $0 < x < L$ and for times $t > 0$ is prescribed by the initial condition:

$$u(x,0) = u_0(x)\,, \qquad (2)$$

and the boundary conditions. In the planes at $x = 0$ and $x = L$, and for times $t > 0$ the condition is one of the following types:

$$\begin{aligned} u &= U_i(t) \\ \frac{\partial u}{\partial n_i} &= f_i(t) \\ \frac{\partial u}{\partial n_i} + Bi\,u &= w_i(t) \end{aligned} \qquad (3)$$

where $U_i(t)$ denotes a surface temperature, $f_i(t)$ and $w_i(t)$ given functions, *Bi* the Biot number, and $\partial/\partial n_i$ differentiation along the outward-drawn normal at the surface *i*. Equations (3) represent the linear first, second, and third kind of boundary conditions (Özisik 1968).

Equations (1), (2), and (3) prescribe also a solution for a semi-infinite and multi-layer slab. In the former region the solution must remain bounded as *x* approaches infinity. In the multi-layer slab a separate solution is required for each layer, and also the continuity of temperatures and heat flows at each interface must be taken account, in addition to the boundary conditions.

The above equations define mathematically the initial and boundary value problem of the finite, semi-infinite, and multi-layer regions. The following pages present an approximate solution for the same problem using the differential-difference approach.

Observe that the following differential equations, initial and boundary conditions are not presented in terms of dimensionless variables. This makes the physical significance of each variable clearer.

2.2 Modification of the differential equation

If the time derivative is written as a backward-difference expression, the original equation becomes a differential-difference equation:

$$\frac{d^2u(x,t)}{dx^2} = RC\frac{u(x,t) - u(x,t-\eta)}{h} + g(x,t), \tag{4}$$

where the denominator h is a constant, and the time variable t gets only discrete values.

The above equation is an ordinary differential-difference equation which may be treated at least in six different ways. Both Bateman (1943) and Pinney (1958) give a survey of the available methods. Typical solutions are also presented by Bellman & Cooke (1963). Rektorys (1982) presents a procedure where he converts the initial and boundary value problem into a solution of p ordinary differential equations with corresponding boundary conditions. He divides the time interval of interest into p subintervals. As a result, he gets p equations of type (4) and p solutions, one for each time subinterval. The following approach to solve Equation (4) differs from those above.

2.3 The complete solution for one time step

When Equation (4) is written as

$$\frac{d^2u(x,t)}{dx^2} - \frac{RC}{h}u(x,t) = -\frac{RC}{h}u(x,t-\eta) + g(x,t), \tag{5}$$

the whole right hand side can be interpreted as a forcing function. The equation represents a non-homogeneous second order differential equation with constant coefficients. From mathematical point of view, x is the only independent variable. The solution is obtained with the method of variation of parameters as

$$u(x,t) = A_0 \cosh x + B_0 \sinh x - q\int_0^x \sinh(x-\nu)u(\nu,t-\eta)\,d\nu + q\int_0^x \sinh(x-\nu)\,g(\nu,t)\,d\nu, \quad (6)$$

where

$$q = \sqrt{\frac{RC}{h}} \quad (7)$$

The temperature function *u(x,t)* represents a complete solution of the above differential equation for the first time step. A particular solution requires suitable values to be assigned to the arbitrary constants A_0 and B_0.

Observe that Equation (6) compared with the corresponding Laplace transform solution in *t*-space of (1) is exactly the same if *h* is replaced by 1/*s* and $u(x,t-n)$ by $u(x,0)$.

2.4 The complete solution for all times

The following procedure changes the complete solution (6) into a series of hyperbolic functions. Each of them represents the temperature at the past time instant. The solution is obtained by substituting *u(x,t)* (6), repeatedly back in the equation. The first integrand of (6) contains the expression $u(x,t-\eta)$, which represents the temperature at the past time instant $t-\eta$, the second at $t-\xi$, $\xi > \eta$, etc. For brevity the current and past time steps are written as *t*, *t* – 1, *t* –2, … The exact value of the time step is not known but one may initially assume that the time step equals the lumping constant *h*. Then, the time derivative expression in (4) becomes a typical backward-difference approximation. The relationship between *h* and η is later discussed in more detail. Observe that the time variable *t* or its past values are not included in the resulting hyperbolic expressions, i.e. the time variable disappears in the particular solution. Thus, the attached time variable *t* – *n*, or actually the subscript *n* = 0, 1, 2, …, only keeps track of the corresponding time step.

Laplace-transformation of (6) gives

$$U(p,t) = A_0\frac{p}{p^2-q^2} + B_0\frac{q}{p^2-q^2} - \frac{q^2}{p^2-q^2}U(p,t-1) + \frac{q^2}{p^2-q^2}G(p,t) \quad (8)$$

One time step later, the corresponding expression $u(x,t-1)$ is

$$U(p,t-1) = A_1 \frac{p}{p^2-q^2} + B_1 \frac{q}{p^2-q^2} - \frac{q^2}{p^2-q^2} U(p,t-2) + \frac{q^2}{p^2-q^2} G(p,t-1)\,. \quad (9)$$

By inserting $U(p,t-1)$ in the preceding equation, simplifying the equation and repeating the procedure many times, gives

$$\begin{aligned} U(p,t) \quad = \quad & \sum_{n=0}^{\infty} \left[A_n\, p \frac{(-1)^n q^{2n}}{(p^2-q^2)^{n+1}} \; + \; B_n \frac{(-1)^n q^{2n+1}}{(p^2-q^2)^{n+1}} \right. \\ & \left. + \; q \frac{(-1)^n q^{2n+1}}{(p^2-q^2)^{n+1}} G(p,t-n) \right] \end{aligned} \tag{10}$$

If the Laplace-transforms of the expressions of (10) are defined as

$$L\{a_m(x)\} \quad = \quad p \frac{(-1)^m q^{2m}}{\left(p^2-q^2\right)^{m+1}} \tag{11}$$

$$L\{b_m(x)\} \quad = \quad \frac{(-1)^m q^{2m+1}}{\left(p^2-q^2\right)^{m+1}}\,, \tag{12}$$

the hyperbolic functions $a_m(x)$ and $b_m(x)$, $m =$ 0,1,2,... can be presented as

$$\begin{aligned} & a_0(x) = \cosh qx \\ & a_1(x) = -\frac{qx}{2} \sinh qx \\ & a_2(x) = -\frac{qx}{4}\left[\frac{1}{2}\sinh qx - \frac{1}{2} qx \cosh qx\right] \\ & \quad \vdots \\ & a_m(x) = -\frac{qx}{2m}\left[\mu_{m,1} \sinh qx - \mu_{m,2}\, qx \cosh qx + \cdots + \mu_{m,m} \frac{(qx)^{m-1}}{(m-1)!} \sinh qx\right] \end{aligned} \tag{13}$$

and

$$\begin{aligned} & b_0(x) = \sinh qx \\ & b_1(x) = \frac{1}{2}\sinh qx - \frac{1}{2} qx \cosh qx \\ & b_2(x) = \frac{3}{2^3}\sinh qx - \frac{3}{2^3} qx \cosh qx + \frac{1}{2^2}\frac{(qx)^2}{2!}\sinh qx \\ & \quad \vdots \\ & b_m(x) = \mu_{m+1,1} \sinh qx - \mu_{m+1,2}\, qx \cosh qx + \cdots + \mu_{m+1,m+1} \frac{(qx)^m}{m!} \sinh qx \end{aligned} \tag{14}$$

Observe that the expressions $a_m(x)$ and $b_m(x)$ of (13) and (14) are presented only for odd numbers of m. The numerical value of the constant $\mu_{m,m}$ depends on the time instant and its position in the expression.

Finally, the complete solution of Equation (6) is obtained with the above definitions for $a_n(x)$ and $b_n(x)$:

$$u(x,t) \;=\; \sum_{n=0}^{\infty}\left[A_n a_n(x) \;+\; B_n b_n(x) \;+ q\int_0^x b_n(x-s)\, g(s,t-n)\, ds \right]. \qquad (15)$$

2.5 Recursive form of the coefficient functions

A closer look into the coefficient functions $a_n(x)$ and $b_n(x)$ reveals that they can be written in a more compact form with the Modified Spherical Bessel functions of the first kind. The first three of those Bessel functions are

$$\begin{aligned}
\sqrt{\frac{\pi}{2v}}I_{\frac{1}{2}}(v) &= \frac{\sinh v}{v} \\
\sqrt{\frac{\pi}{2v}}I_{\frac{3}{2}}(v) &= -\frac{\sinh v}{v^2}+\frac{\cosh v}{v} \\
\sqrt{\frac{\pi}{2v}}I_{\frac{5}{2}}(v) &= \left(\frac{3}{v^2}+\frac{1}{v}\right)\sinh v-\frac{3}{v^2}\cosh v
\end{aligned} \qquad (16)$$

By comparing Equations (13), (14) and (16), an expression for $b_n(x)$ can be written as

$$b_n(x) \;=\; \sqrt{\frac{\pi}{2}}(-1)^n \frac{(qx)^{n+1/2}}{n!2^n} I_{n+1/2}(qx) \qquad (17)$$

If the following recurrence relations of the Modified Spherical Bessel functions are applied:

$$I_{n+3/2}(v) \;=\; I_{n-1/2}(v)-\frac{2n+1}{v}I_{n+1/2}(v), \qquad (18)$$

$$\frac{d}{dv}\left[\sqrt{\frac{\pi}{2v}}I_{n+1/2}(v)\right] \;=\; \sqrt{\frac{\pi}{2v}}\left[I_{n-1/2}(v)-\frac{n+1}{v}I_{n+1/2}(v)\right] \qquad (19)$$

the coefficient functions can be easily put in recursive form. The procedure and formulas of $a_n(x)$, $b_n(x)$ and their derivatives are presented in Table 1. A computer program based on the procedure needs only a few lines to compute the value of coefficients.

Table 1. A procedure for computing the coefficient functions.

$a_n(x)$ **and** $b_n(x)$	$a'_n(x)$ **and** $b'_n(x)$
1. Calculate initial values $a_0(x) = \cosh qx$ $b_0(x) = \sinh qx$	1. Calculate initial values $\frac{1}{q}a_0'(x) = \sinh qx$ $\frac{1}{q}b_0'(x) = \cosh qx$
2. Set $n = 1$	2. Set $n = 1$
3. Calculate functions $a_n(x) = -\frac{qx}{2n}b_{n-1}(x)$ $b_n(x) = \frac{1}{2n}\left[(2n-1)b_{n-1}(x) - qx\, b'_{n-1}(x)\right]$	3. Calculate functions $\frac{1}{q}a_n'(x) = -\frac{1}{2n}\left[qx\, b'_{n-1}(x) + b_{n-1}(x)\right]$ $\frac{1}{q}b_n'(x) = a_n(x)$
4. Set $n = n + 1$	4. Set $n = n + 1$
5. if $n < n_{\max}$ go to step 3.	5. if $n < n_{\max}$ go to step 3.

3 Conduction of heat through a finite, one-layer slab

3.1 The zeroth order transfer function; Case 1

A particular solution is obtained from the complete solution (15), when arbitrary constants A_n, B_n, n = 0, 1, 2,... are determined. Consider heat conduction through a slab of thickness L which is initially at zero temperature, and no inner heat is produced. The boundary conditions are

$$
\begin{aligned}
&1)\ u = 0, \text{ at } x = 0,\ t > 0 \\
&2)\ u = U_2(t-n), \text{ at } x = L,\ t > 0,\ n = 0, 1, 2, \ldots
\end{aligned} \tag{20}
$$

Observe the difference with the corresponding analytical boundary conditions. Instead of the analytical temperature function $U_2(t)$, only its sampled values at discrete time instants are applied.

According to (6) the complete solution at the time instant $t - n$ is

$$u(x,t-n) = A_n \cosh x + B_n \sinh x + J(x,t-n-1) \tag{21}$$

where $J(x,t-n-1)$ refers to the first integral term from the left of Equation (6). Substituting the boundary conditions makes possible to write the arbitrary constants A_n and B_n as

$$
\begin{aligned}
A_n &= 0 \\
B_n &= \frac{U_2(t-n)}{\sinh qL} - \frac{J(L,t-n-1)}{\sinh qL}
\end{aligned} \tag{22}
$$

Inserting these in the complete solution of (15), gives

$$
\begin{aligned}
u(x,t) &= b_{1,0}(x)\big[U_2(t) - J(L,t-1)\big] + b_{1,1}(x)\big[U_2(t-1) - J(L,t-2)\big] \\
&+ b_{1,2}(x)\big[U_2(t-2) - J(L,t-3)\big] + \cdots,
\end{aligned} \tag{23}
$$

In (23) the hyperbolic function $b_{1,n}(x)$ is defined as

$$b_{1,n}(x) = \frac{b_n(x)}{\sinh qL} \tag{24}$$

where the subscript n refers to time step t - n.

Substituting $u(x,t)$ from Equation (23) in the second boundary condition with $n = 0$, allows one to solve the first integral term $J(L,t-1)$:

$$\begin{aligned} J(L,t-1) \;=\; & b_{1,1}\left[U_2(t-1) - J(L,t-2)\right] + b_{1,2}\left[U_2(t-2) - J(L,t-3)\right] \\ & + \; b_{1,3}\left[U_2(t-3) - J(L,t-4)\right] + \cdots, \end{aligned} \tag{25}$$

where $b_{1,n}$ means a shorter form of $b_{1,n}(L)$. After inserting $J(L,t-1)$ back in Equation (23), and making some rearrangements, $u(x,t)$ can be rewritten as

$$\begin{aligned} u(x,t) = \; & b_{1,0}(x)\,U_2(t) + b_{2,1}(x)\,U_2(t-1) + b_{2,2}(x)\,U_2(t-2) + \cdots \\ & - b_{2,1}(x)\,J(L,t-2) - b_{2,2}(x)\,J(L,t-3) - \cdots \end{aligned} \tag{26}$$

The above notation for $b_{2,i}(x)$, $i = 1,2,3,...$, means

$$\begin{aligned} b_{2,1}(x) \;&=\; b_{1,1}(x) \;-\; b_{1,0}(x)b_{1,1} \\ b_{2,2}(x) \;&=\; b_{1,2}(x) \;-\; b_{1,0}(x)b_{1,2} \\ &\;\vdots \\ b_{2,n}(x) \;&=\; b_{1,n}(x) \;-\; b_{1,0}(x)b_{1,n} \end{aligned} \tag{27}$$

Next, the integral term $J(x,t-2)$ must be removed from (26). This is done in the following way. First, $J(x,t-2)$ is solved from Equation (25) by subtracting all time instants by one and then, the resulting $J(x,t-2)$ is inserted in (26). This creates a new expression for $u(x,t)$ with one fewer integral term. After rearranging, all but the first two hyperbolic functions in Equation (26) have changed. Others are denoted as $b_{3,i}(x)$, i=1,2,3,..., and their structure is similar to Equation (27). By repeating the procedure all integral terms vanish, and the final solution is written as

$$\begin{aligned} u(x,t) \;\;&=\;\; b_{1,0}(x)U_2(t) + b_{2,1}(x)U_2(t-1) + b_{3,2}(x)U_2(t-2) \\ &\qquad + b_{4,3}(x)U_2(t-3) + b_{5,4}(x)U_2(t-4) + \cdots \\ &=\;\; \sum_{n=0}^{\infty} b_{n+1,n}(x)U_2(t-n), \end{aligned} \tag{28}$$

where $b_{1,0}(x)$is obtained from (24) and the all other functions $b_{n+1,n}(x)$ from

$$b_{m,n}(x) \;=\; b_{m-1,n}(x) \;-\; b_{m-1,m-2}(x)b_{1,n-m+2}\,; \;\; m \;=\; 2,3,4,\cdots; \;\; n \;\geq\; m-1. \tag{29}$$

Observe that the hyperbolic function $b_{n+1,n}(x)$ of (28) is actually a transfer functions which determines response of $u(x,t)$ to an exciting surface temperature U_2 at the specified time instant t - n.

3.2 The zeroth order transfer function; Case 2

If the boundary conditions of the previous problem are changed to

$$\begin{aligned}
&1)\ u\ =\ U_1(t-n),\ \text{at } x = 0,\ t\ > 0,\ n = 0,\ 1,\ 2,\ldots \\
&2)\ u\ =\ 0,\ \text{at } x = L,\ t\ > 0
\end{aligned} \tag{30}$$

the particular solution is obtained similarly. Substituting the boundary conditions generates the following arbitrary constants

$$\begin{aligned}
A_n\ &=\ U_1(t-n) \\
B_n\ &=\ \frac{\cosh qL}{\sinh qL}U_1(t-n)\ -\ \frac{J(L,t-n-1)}{\sinh qL}
\end{aligned} \tag{31}$$

When they are inserted in Equation (31), the solution will be

$$\begin{aligned}
u(x,t)\ =\ & a_{1,0}(x)U_1(t)-b_{1,0}(x)\,J(L,t-1)\ +\ a_{1,1}(x)U_1(t-1) \\
& -b_{1,1}(x)\,J(L,t-1)+\cdots,
\end{aligned} \tag{32}$$

In this equation $b_{1,n}(x)$ is defined by (24) and $a_{1,n}(x)$ by (33)

$$a_{1,n}(x)\ =\ a_n(x)\ -\ \frac{\cosh qL}{\sinh qL}b_n(x), \tag{33}$$

where n = 0,1,2,.... Substituting the second boundary condition and proceeding as before, the particular solution is finally written as

$$\begin{aligned}
u(x,t)\ &=\ a_{1,0}(x)U_1(t)+a_{2,1}(x)U_1(t-1)+a_{3,2}(x)U_1(t-2) \\
&\quad +a_{4,3}(x)U_1(t-3)+a_{5,4}(x)U_1(t-4)+\cdots \\
&=\ \sum_{n=0}^{\infty}a_{n+1,\,n}(x)U_1(t-n),
\end{aligned} \tag{34}$$

where

$$a_{m,n}(x)\ =\ a_{m-1,\,n}(x)\ -\ b_{m-1,\,m-2}(x)\,a_{1,\,n-m+2}\,;\ m\ =\ 2,3,4,\cdots;\ n\ \geq\ m-1 \tag{35}$$

The resulted solution (34), together with the solution (28), is later applied in summing the coefficient functions and in developing heat conduction through a multi-layer wall.

3.3 The transfer function of k^{TH} order

The particular solution of the foregoing chapter were all presented as a zeroth order transfer function. If the zeroth order transfer function is used in practice, a number of coefficient functions must be included. This is due to the fact that series solutions of $a_{n+1,n}(x)$ and $b_{n+1,n}(x)$ are not rapidly converging. The number of coefficient functions can be reduced if $u(x,t)$ at past time instants becomes part of the solution. Similarly to the pulse transfer method, a zeroth order transfer function can be changed to a k^{th}-order transfer function (Hittle 1982), where the value k refers to the number of past time steps. However, the following technique is different from Hittle.

Consider the solution (28):

$$\begin{aligned} u(x,t) \; = \; & b_{1,0}(x)U_2(t) + b_{2,1}(x)U_2(t-1) + b_{3,2}(x)U_2(t-2) \\ & + b_{4,3}(x)U_2(t-3) + b_{5,4}(x)U_2(t-4) + \cdots \end{aligned} \tag{36}$$

If $b_{1,1}\,u(x,t-1)$ is added to both sides of the equation, then

$$\begin{aligned} u(x,t) + b_{1,1}u(x,t-1) \; = \; & b_{1,0}(x)U_2(t) + \left[b_{2,1}(x) + b_{1,0}(x)b_{1,1}\right]U_2(t-1) \\ & + \left[b_{3,2}(x) + b_{2,1}(x)b_{1,1}\right]U_2(t-2) + \left[b_{4,3}(x) + b_{3,2}(x)b_{1,1}\right]U_2(t-3) + \cdots \end{aligned} \tag{37}$$

When the expressions inside the brackets are compared with Equations (27) and (29), Equation (37) can be reduced to

$$\begin{aligned} u(x,t) + b_{1,1}u(x,t-1) \; = \; & b_{1,0}(x)U_2(t) + b_{1,1}(x)U_2(t-1) \\ & + b_{2,2}(x)U_2(t-2) + b_{3,3}(x)U_2(t-3) + \cdots \end{aligned} \tag{38}$$

Equation (38) can be further simplified by adding $b_{1,2}\,u(x,t-2)$ to both sides of the equation. After repeating this procedure k times, a k^{th}-order transfer function is produced. In practice, numerical values of $u(x,t)$ are computed using k^{th}-order transfer functions. In theory, k may also approach infinity. Then, the solution is written as

$$u(x,t) \; = \; -\sum_{n=1}^{\infty} b_{1,n}\,u(x,t-n) \; + \; \sum_{n=0}^{\infty} b_{1,n}(x)U_2(t-n) \tag{39}$$

The above solution represents a transfer function of infinite order, where the right-hand-side of the first boundary condition of (20) remains zero. If both boundary conditions are non-zero, the transfer function of infinite order is achieved as previously, starting from the sum of Equations (34) and (28).

Another choice is to use the superposition principle and solutions of the form (39). Therefore, using the boundary conditions:

1) $u = U_1(t-n)$, at $x = 0$, $t > 0$, $n = 0, 1, 2, \ldots$

2) $u = U_2(t-n)$, at $x = L$, $t > 0$, $n = 0, 1, 2, \ldots$ (40)

the solution in the form of infinite order is obtained as

$$u(x,t) = -\sum_{n=1}^{\infty} b_{1,n}\, u(x,t-n) + \sum_{n=0}^{\infty} c_{1,n}(x) U_1(t-n) + \sum_{n=0}^{\infty} b_{1,n}(x) U_2(t-n) \qquad (41)$$

where

$$\begin{aligned}
c_{1,0}(x) &= a_{1,0}(x) \\
c_{1,1}(x) &= a_{2,1}(x) + b_{1,1}\, a_{1,0}(x) \\
c_{1,2}(x) &= a_{3,2}(x) + b_{1,1}\, a_{2,1}(x) + b_{1,2}\, a_{1,0}(x) \\
&\vdots \\
c_{1,n}(x) &= a_{n+1,n}(x) + b_{1,1}\, a_{n,n-1}(x) + \cdots + b_{1,n}\, a_{1,0}(x)
\end{aligned} \qquad (42)$$

The above cases represented only solutions to the first kind of boundary conditions. The second and third kind of boundary condition is applied similarly. Some examples are illustrated later in chapters 4 and 5.

3.4 Sum of the coefficient functions

In a steady state the coefficient functions form a convergent, functional series. The sum of the series is related to the steady state value of the temperature. If the sum of the coefficient functions is close enough to the specified steady state value, the effect of truncation error can be neglected. Thus, summing of the coefficient functions is a convenient way to check a proper number of those functions, in order to achieve the targeted accuracy of the numerical results.

This procedure is outlined using the solution *u(x,t)*, subject to the boundary conditions (40) but presented by means of the zeroth order transfer function. This is achieved using the superposition principle and summing the solutions (28) and (34):

$$u(x,t) = \sum_{n=0}^{\infty} a_{n+1,n}(x) U_1(t-n) + \sum_{n=0}^{\infty} b_{n+1,n}(x) U_2(t-n) \qquad (43)$$

If all values of $U_1(t)$ and $U_2(t)$ remain constant, sums of the series formed by the coefficient functions can be calculated. Comparing solutions of Equations (41) and (43) in steady state, the following equalities can be written immediately:

$$\sum_{n=0}^{\infty} a_{n+1,n}(x) = \frac{\sum_{n=0}^{\infty} c_{1,n}(x)}{\sum_{n=0}^{\infty} b_{1,n}} \tag{44}$$

and

$$\sum_{n=0}^{\infty} b_{n+1,n}(x) = \frac{\sum_{n=0}^{\infty} b_{1,n}(x)}{\sum_{n=0}^{\infty} b_{1,n}} \tag{45}$$

In a steady state the temperature inside a slab is a straight line connecting the surface at temperatures U_1 and U_2:

$$\lim_{t \to \infty} u(x,t) = U_1 - (U_1 - U_2)x \tag{46}$$

Again comparing this with the solutions of $u(x,t)$, one can conclude that

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n+1,n}(x) &= \sum_{n=0}^{\infty} c_{1,n}(x) = 1 - x \\ \sum_{n=0}^{\infty} b_{n+1,n}(x) &= \sum_{n=0}^{\infty} b_{1,n}(x) = x \end{aligned} \tag{47}$$

Convergence of the series solution is more closely examined in appendix 1.

4 Conduction of heat through a semi-infinite, one-layer slab

4.1 The coefficient functions in semi-infinite region

Similarly to the finite region, the complete solution (15) can be applied to heat conduction problems in semi-infinite region. The solution must be bounded as x tends to infinity. Therefore, the hyperbolic functions $a_n(x)$ and $b_n(x)$ are reduced to exponential functions. While the hyperbolic functions are closely related to the Modified Bessel functions of the first kind, the exponential functions can be described using the Modified Bessel functions of the third kind. Thus, correspondingly to (13) and (14) the coefficient functions $a_n(x)$ are written as

$$
\begin{aligned}
a_0(x) &= e^{-qx} \\
a_1(x) &= \frac{1}{2} qx\, e^{-qx} \\
a_2(x) &= \frac{1}{2^3} qx(1 + qx)\, e^{-qx} \\
&\vdots \\
a_n(x) &= \frac{qx}{2n}\left[\mu_{n,1} + \mu_{n,2}\, qx + \mu_{n,3}\frac{(qx)^2}{2!} + \cdots + \mu_{n,n}\frac{(qx)^{n-1}}{(n-1)!}\right] e^{-qx}
\end{aligned}
\tag{48}
$$

and $b_n(x)$ as

$$
\begin{aligned}
b_0(x) &= -\, e^{-qx} \\
b_1(x) &= -\frac{1}{2}(1 + qx)\, e^{-qx} \\
b_2(x) &= -\frac{1}{2^3}\left[3 + 3qx + (qx)^2\right] e^{-qx} \\
&\vdots \\
b_n(x) &= \left[\mu_{n+1,1} + \mu_{n+1,2}\, qx + \mu_{n+1,3}\frac{(qx)^2}{2!} + \cdots + \mu_{n+1,n+1}\frac{(qx)^n}{n!}\right] e^{-qx}
\end{aligned}
\tag{49}
$$

The procedure for computing the coefficient functions presented in Table 1 is still valid, if the initial coefficients are replaced with those of (48) and (49). Observe that particular solutions in the semi-infinite region are straight-forward, because the integral function $J(x,t-n)$ equals zero on the boundary of the slab and has no effect on the structure of the solution.

4.2 Boundary conditions of the first kind

Consider a semi-infinite slab. Its temperature is initially at zero, no heat is produced inside the solid, and the surface temperature $U(t)$ is varying with time. The boundary condition is therefore

$$u = U(t-n)\text{, at } x = 0,\ t > 0,\ n = 0, 1, 2, \ldots \tag{50}$$

The complete solution for the homogeneous case is written as

$$u(x,t) = \sum_{n=0}^{\infty}\left[A_n a_n(x) + B_n b_n(x)\right] \tag{51}$$

When the boundary condition is inserted in (51) all $b_n(0) = 0$ and $a_n(0) = 0$, $n = 0, 1, 2, \ldots$, except $a_0(0) = 1$ resulting that $A_0 = U(t)$. Obviously A_1, A_2, A_3, … are also non-zero. Therefore, one can make a guess that

$$A_n = U(t-n) \tag{52}$$

As a result, the particular solution can be written immediately as

$$u(x,t) = \sum_{n=0}^{\infty} a_n(x)U(t-n) \tag{53}$$

If the solution is written using the Modified Spherical Bessel functions of the third kind the mathematical expression will be

$$u(x,t) = \sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{(qx)^{n+\frac{1}{2}}}{n!2^n}K_{n-\frac{1}{2}}(qx)U(t-n) \tag{54}$$

An interesting special case is a semi-infinite slab, in which the temperature is initially at zero, no inner heat is generated, and the solid is heated by a flux $Q(t)$ in the plane $x = 0$. The boundary condition is therefore

$$-k\frac{du}{dx} = Q(t-n)\text{, } x = 0,\ t > 0,\ n = 0, 1, 2, \ldots \tag{55}$$

Inserting the boundary condition and proceeding similarly as before the solution will be

$$u(x,t) = -\frac{1}{kq}\sum_{n=0}^{\infty} b_n(x)Q(t-n) \tag{56}$$

Especially in the plane $x = 0$, when Equation (7) is applied the first expressions of the solution are

$$u(0,t) = R\sqrt{\frac{h}{RC}}\left[Q(t) + \frac{1}{2}Q(t-1) + \frac{3}{2^2}Q(t-2) + \frac{10}{2^5}Q(t-3) + \cdots\right]. \quad (57)$$

4.3 Boundary conditions of the third kind

Consider a semi-infinite slab, initially at zero temperature. The solid is heated by radiation in the plane $x = 0$ from a medium at $V(t)$, and no inner heat is produced. Then, the boundary condition is

$$k\frac{du}{dx} = h_S\left[u - V(t-n)\right],\ x = 0,\ t > 0,\ n = 0, 1, 2, \ldots \quad (58)$$

where h_s denotes convective heat transfer coefficient and $V(t - n)$ the room temperature. The complete solution is equal to Equation (51). When the constant A_n and B_n are solved only B_n generates a solution satisfying (58). Thus, the arbitrary constant is

$$B_n = -\frac{h_S}{kq + h_S}V(t-n) \quad (59)$$

As a result, the particular solution is

$$u(x,t) = -\frac{h_S}{k\sqrt{\frac{RC}{h}} + h_S}\sum_{n=0}^{\infty} b_n(x)V(t-n) \quad (60)$$

5 Conduction of heat through a multi-layer slab or wall

5.1 Definition of two-port parameters

Two-port parameters are transfer functions describing relationships between heat flow rates and temperatures on both sides of the wall. These parameters regard the wall or slab as a black box where only its inputs and outputs are accessible. Two-port parameters form a convenient way to present transfer functions for cascaded systems such as multi-layer walls. The following pages show how the differential-difference solution is presented using the two-port parameters.

Consider Equation (41)

$$u(x,t) = -\sum_{n=1}^{\infty} b_{1,n}\, u(x,t-n) + \sum_{n=0}^{\infty} c_{1,n}(x) U_1(t-n) + \sum_{n=0}^{\infty} b_{1,n}(x) U_2(t-n) \qquad (61)$$

By defining $P_1(t - n)$ and $P_2(t - n)$, $n = 0, 1, 2, \dots$ as

$$P_1(t-n) = -k\frac{du}{dx}, \text{ at } x = 0,\ t > 0 \qquad (62)$$

$$P_2(t-n) = -k\frac{du}{dx}, \text{ at } x = L,\ t > 0 \qquad (63)$$

Then, differentiating (61) and inserting the derivatives (62) and (63) the heat flow rates $P_1(t)$ and $P_2(t)$ can be solved. Observe also that inserting $x = 0$ or $x = L$ simplifies several coefficient functions, such as

$$\begin{aligned}
b_{1,0}(0) &= \frac{qL}{\sinh qL} \\
b_{1,n}(0) &= 0,\ n \ge 1 \\
c'_{1,n}(0) &= -\frac{qL\, a_n}{\sinh qL},\ n \ge 0 \\
c'_{1,0}(L) &= -\frac{qL}{\sinh qL} \\
c'_{1,n}(L) &= 0,\ n \ge 1
\end{aligned} \qquad (64)$$

Finally, the transfer function consists of the pair of equations

$$b_0 \frac{1}{kq} P_1(t) \ + \ b_1 \frac{1}{kq} P_1(t-1) \ + \ b_2 \frac{1}{kq} P_1(t-2) \ + \cdots$$
$$= -U_2(t) \ + \ a_0 U_1(t) \ + \ a_1 U_1(t-1) \ + \ a_2 U_1(t-2) \ + \cdots \tag{65}$$

$$b_0 \frac{1}{kq} P_2(t) \ + \ b_1 \frac{1}{kq} P_2(t-1) \ + \ b_2 \frac{1}{kq} P_2(t-2) \ + \cdots$$
$$= U_1(t) \ - \ b'_0 U_2(t) \ - \ b'_1 U_2(t-1) \ - \ b'_2 U_2(t-2) \ + \cdots \tag{66}$$

Defining a backshift operator

$$z^n U(t) \ = \ U(t-n) \tag{67}$$

Equations (65) and (66) can be further written as

$$(b_0 + b_1 z + b_2 z^2 + \cdots) \frac{1}{kq} P_1(t) \ = \ (a_0 + a_1 z + a_2 z^2 + \cdots) U_1(t) \ - \ U_2(t) \tag{68}$$

$$(b_0 + b_1 z + b_2 z^2 + \cdots) \frac{1}{kq} P_2(t) \ = \ U_1(t) \ - \ (b'_0 + b'_1 z + b'_2 z^2 + \cdots) U_2(t) \ . \tag{69}$$

If the following notations

$$\begin{aligned}
A(z) \ &= \ (b'_0 + b'_1 z + b'_2 z^2 + \ b'_3 z^3 \ + \ \cdots) q^{-1} L^{-1} \\
B(z) \ &= \ (b_0 + b_1 z + b_2 z^2 + \ b_3 z^3 \ + \ \cdots)(kq)^{-1} \\
C(z) \ &= \ (a'_0 + a'_1 z + a'_2 z^2 + \ a'_3 z^3 \ + \ \cdots) kL^{-1} \\
D(z) \ &= \ a_0 + a_1 z + a_2 z^2 + \ a_3 z^3 \ + \ \cdots
\end{aligned} \tag{70}$$

are adopted Equations (68) and (69) can be written in matrix form as

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \frac{1}{B(z)} \begin{bmatrix} D(z) & -1 \\ 1 & -A(z) \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix}. \tag{71}$$

5.2 Selection of two-ports

Equation (71) shows one possibility of different combinations of two-port matrices which relate the heat flow rates and temperatures on both side of a slab. The rest of the expressions are:

$$\begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \frac{1}{C(z)} \begin{bmatrix} A(z) & -1 \\ 1 & -D(z) \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix}$$

$$\begin{bmatrix} P_1(t) \\ U_2(t) \end{bmatrix} = \frac{1}{A(z)} \begin{bmatrix} C(z) & -1 \\ 1 & -B(z) \end{bmatrix} \begin{bmatrix} U_1(t) \\ P_2(t) \end{bmatrix}$$

$$\begin{bmatrix} U_1(t) \\ P_2(t) \end{bmatrix} = \frac{1}{D(z)} \begin{bmatrix} B(z) & -1 \\ 1 & -C(z) \end{bmatrix} \begin{bmatrix} P_1(t) \\ U_2(t) \end{bmatrix} \tag{72}$$

$$\begin{bmatrix} U_1(t) \\ P_1(t) \end{bmatrix} = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \begin{bmatrix} U_2(t) \\ P_2(t) \end{bmatrix}$$

The last matrix equation is called the transmission matrix. It is needed when conduction of heat is computed through a multi-layer wall. The determinant of the transmission matrix is one i.e.

$$A(z)D(z) + B(z)C(z) = 1 \tag{73}$$

This can be verified using expressions (70).

5.3 Cascaded two-ports

Boundary conditions of the third kind

Cascaded two-ports combine, not only the layers of the wall but also their surface areas. The following example shows how the conditions of the surface are included in the two-port presentation. Consider a homogeneous, finite, one-layer slab, initially at zero uniform temperature which is subjected to the following boundary conditions

1) $-k\dfrac{du}{dx} = h_S U_1(t-n)$, at $x = 0$, t > 0

2) $u = U_2(t-n)$, at $x = L$, $t > 0$ (74)

and $n = 0, 1, 2, \ldots$ If the solution is modified to a matrix notation, it will be

$$\begin{bmatrix} 0 \\ P_1(t) \end{bmatrix} = \begin{bmatrix} 1 & h_S^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \begin{bmatrix} U_2(t) \\ P_2(t) \end{bmatrix}. \tag{75}$$

Using ordinary matrix multiplication, a pair of equations is generated. An unnecessary variable can be removed by direct substitution from the other equation. For example, if $P_1(t)$ is solved as a function of $U_1(t)$ and $U_2(t)$ the expression is written as

$$P_1(t) = \frac{D(z)}{B(z)+h_S^{-1}D(z)} U_1(t) - \frac{1}{B(z)+h_S^{-1}D(z)} U_2(t) \tag{76}$$

Inserting power series *B(z)* and *D(z)* finally gives

$$P_1(t) = -\frac{1}{b_0 + h_S^{-1}}\left[\sum_{n=1}^{\infty}(b_n + h_S^{-1} a_n) P_1(t-n) - \sum_{n=0}^{\infty} a_n U_1(t-n) - U_2(t)\right] \quad (77)$$

A multi-layer wall

If the outer surface temperatures and heat flow rates of a multi-layer wall are denoted as $U_1(t)$, $P_1(t)$, $U_{n+1}(t)$ and $P_{n+1}(t)$, relationships between them can be written as

$$\begin{bmatrix} U_1(t) \\ P_1(t) \end{bmatrix} = \begin{bmatrix} A_1(z) & B_1(z) \\ C_1(z) & D_1(z) \end{bmatrix} \begin{bmatrix} A_2(z) & B_2(z) \\ C_2(z) & D_2(z) \end{bmatrix} \cdots \begin{bmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{bmatrix} \begin{bmatrix} U_{n+1}(t) \\ P_{n+1}(t) \end{bmatrix}, \quad (78)$$

where subscripts 1, 2, …, *n* refer to the layers.

A practical way to compute (78) is to process two layers at the time. First, assume that the *n*-layer wall consists of two parts. One of them is the current layer *i* and the other is all the preceding layers *i* – 1, *i* – 2, … In matrix form this is written as

$$\begin{aligned} \begin{bmatrix} U_1(t) \\ P_1(t) \end{bmatrix} &= \begin{bmatrix} W_1(z) & W_2(z) \\ W_3(z) & W_4(z) \end{bmatrix} \begin{bmatrix} V_1(z) & V_2(z) \\ V_3(z) & V_4(z) \end{bmatrix} \begin{bmatrix} U_{n+1}(t) \\ P_{n+1}(t) \end{bmatrix} \\ &= \begin{bmatrix} R_1(z) & R_2(z) \\ R_3(z) & R_4(z) \end{bmatrix} \begin{bmatrix} U_{n+1}(t) \\ P_{n+1}(t) \end{bmatrix} \end{aligned} \quad (79)$$

where

$$W_n = w_{n,0} + w_{n,1} z + w_{n,2} z^2 + \cdots \quad (80)$$

$$V_n = v_{n,0} + v_{n,1} z + v_{n,2} z^2 + \cdots \quad (81)$$

After multiplication and addition, the elements of transmission matrix for Equation (79) are given by

$$
\begin{aligned}
R_1(z) &= \sum_{i=0}^{n}\left[w_{1,i}v_{2,n-i} + w_{2,i}v_{3,n-i}\right]z^n \\
R_2(z) &= \sum_{i=0}^{n}\left[w_{1,i}v_{2,n-i} + w_{2,i}v_{4,n-i}\right]z^n \\
R_3(z) &= \sum_{i=0}^{n}\left[w_{3,i}v_{2,n-i} + w_{4,i}v_{3,n-i}\right]z^n \\
R_4(z) &= \sum_{i=0}^{n}\left[w_{3,i}v_{2,n-i} + w_{4,i}v_{4,n-i}\right]z^n
\end{aligned}
\qquad (82)
$$

The actual solution for multi-layer wall is achieved in the following way. First, coefficients of Equation (82) are calculated for the first two layers. These results represent coefficients of both first and second layer and can be assigned to elements W_1, W_2, W_3, W_4. The next layer is assigned to elements V_1, V_2, V_3, V_4. By repeating this procedure, coefficients over the whole wall structure are obtained. Finally $P_1(t)$ and $P_{n+1}(t)$ are solved from

$$
\begin{bmatrix} P_1(t) \\ P_{n+1}(t) \end{bmatrix} = \frac{1}{W_1V_2 + W_2V_4}\begin{bmatrix} W_3V_2 + W_4V_4 & -1 \\ 1 & -W_1V_2 - W_1V_3 \end{bmatrix}\begin{bmatrix} U_1(t) \\ U_{n+1}(t) \end{bmatrix}. \qquad (83)
$$

6 Time derivative approximation

6.1 Backward-difference approximation

In practice *h* and η are often set equal in Equation (4). With this substitution the time derivative approximation becomes an ordinary backward-difference approximation. The backward-difference derivative approximation comes from Taylor series. Assume an analytical function $u(x,t)$ with continuous variables *x* and *t*. If the function and its time derivative are finite, continuous and single-valued function $u(x,t)$ can be expanded in the form of Taylor series about the point *t* as

$$u(x,t-h) = u(x,t) - h\frac{\partial u(x,t)}{\partial t} + \frac{h^2}{2!}\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{h^3}{3!}\frac{\partial^3 u(x,t)}{\partial t^3} + \cdots \qquad (84)$$

From (84) the time derivative approximation is obtained as

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t) - u(x,t-h)}{h} + O(h) \qquad (85)$$

The last variable $O(h)$ indicates the magnitude of truncation error which is the order of *h*. The truncation error becomes smaller only when *h* approaches zero.

6.2 Limits of the lumping constant

The lumping constant

In order to find another and perhaps more appropriate relationship between the lumping constant *h* and the displacement constant η, some potential values of *h* will be examined. The value of the constant η is assumed to be known.

Consider the time derivative approximation of (4). Assume that the time derivative expression is separated from (4), and then written to a first order differential equation. Finally, *h* is solved. As a result the following equation is produced

$$h = \frac{u(x,t) - u(x,t-\eta)}{\frac{\partial u(x,t)}{\partial t}} \qquad (86)$$

The limits of interest are the following: $t \to \infty$ and $t \to \eta$ which determine the range of allowable values of *h* in time scale. The following pages present one case study concerning the limits.

A finite, one-layer slab

Consider a homogeneous slab, initially at zero temperature, and for times $t > 0$ the surface $x = L$ is subjected to a stepwise temperature change *U*, while boundary surface $x = 0$ remains at zero. An analytical solution of the temperature will be (Carslaw & Jaeger 1959)

$$u(x,t) = Ux + \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{n^2\pi^2 t}{RC}}\sin n\pi x \tag{87}$$

By defining

$$r_t = e^{-\frac{\pi^2 t}{RC}} \text{ and } r_\eta = e^{-\frac{\pi^2 \eta}{RC}},$$

applying Equation (87) in the expression (86), and finally inserting r_t and r_η the lumping constant *h* can be written in the form

$$h = \frac{\frac{1}{\pi}(1-r_\eta)\left[1-(1+r_\eta+r_\eta{}^2+\ r_\eta{}^3)r^3\cos\pi x+\cdots\right]}{-\frac{\pi}{RC}\left[1-4r^3\cos\pi x+\cdots\right]} \tag{88}$$

As $t \to \infty$, *h* approaches the expression

$$h \to RC\frac{e^{\frac{\pi^2\eta}{RC}}-1}{\pi^2} \tag{89}$$

The other limit as $t \to \eta$ is determined similarly. It turns out that the value of *h* approaches zero. Thus, for a step response of one layer slab, *h* may vary in the following region

$$0 < h < RC\frac{e^{\frac{\pi^2\eta}{RC}}-1}{\pi^2} \tag{90}$$

A multi-layer slab

The corresponding limits of h in a multi-layer environment could be determined accordingly. However, in this case such a solution involves somewhat complicated algebra. Therefore the following approximate procedure is applied. If one has a slab of n layers, each layer of equal thickness L and diffusivity κ, their combined time constant RC_{ALL} is related to the time constant of a slab of thickness nL and diffusivity κ. This is achieved approximately if

$$RC_{ALL} = \left\lfloor \sqrt{R_1C_1} + \sqrt{R_2C_2} + \cdots + \sqrt{R_nC_n} \right\rfloor^2, \qquad (91)$$

where R_nC_n presents a time constant of layer n. Thus, the expression (90) is applicable also to a multi-layer slab, if RC is replaced by RC_{ALL}.

6.3 Value of the lumping constant

Equation (90) defines approximate limits for the lumping constant, both for a one-layer and a multi-layer slab. As a default h and η are set equal. Another alternative would be the average of the limits (90). Actually, the optimized value for h depends on several variables and boundary conditions and its mathematical expression is complicated. However, the test runs indicate that in the above case, a suitable value of h is greater than one time step but still lower than the upper limit of (90). Therefore, one suggestion is the proposal presented by Equation (92).

$$h = \frac{1}{2}\left[\eta + RC\frac{e^{\frac{\pi^2\eta}{RC}} - 1}{\pi^2}\right] \qquad (92)$$

Figures 1 and 2 compare the responses when h is set equal both to η and the expression (92). In the figures the former is referred as the conventional and the latter as the new value of h.

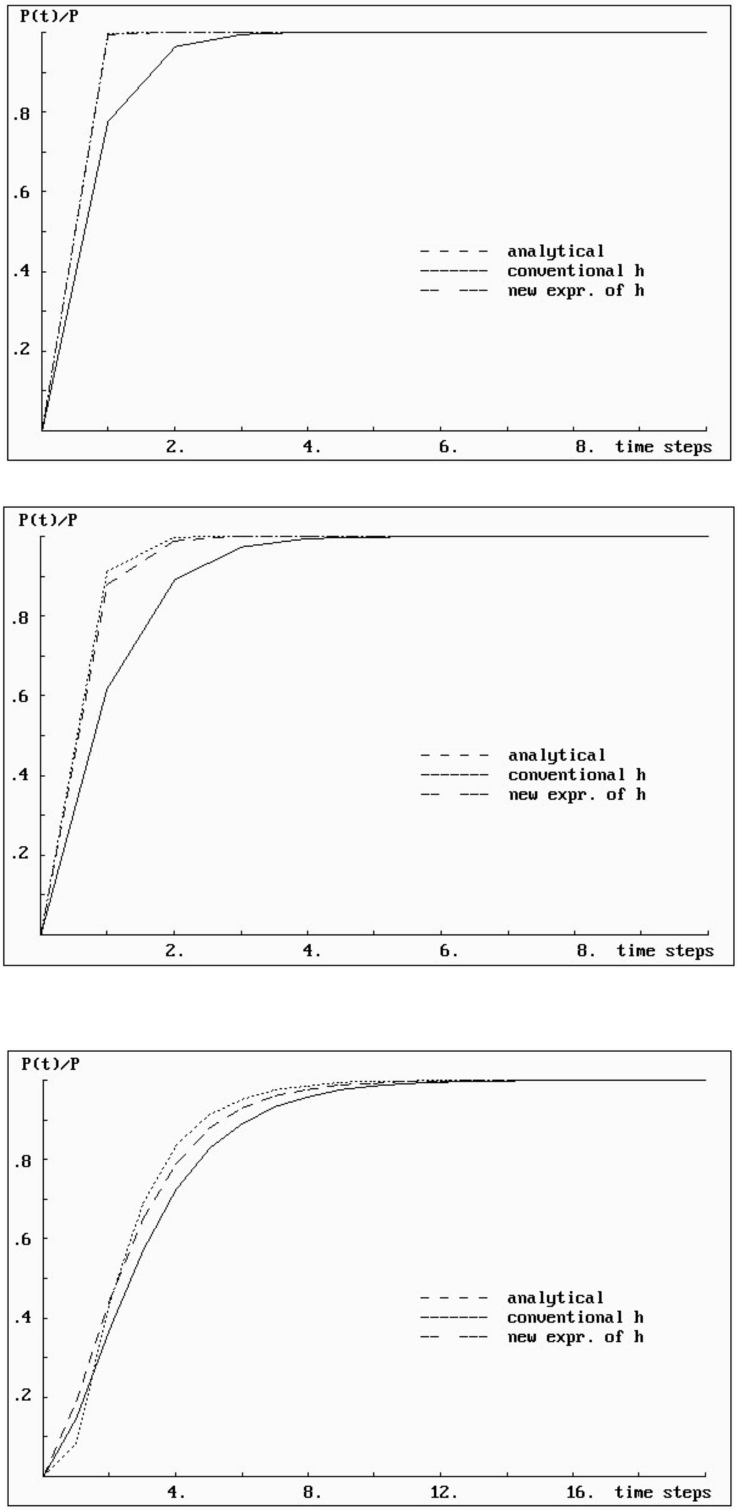


Figure 1. Comparison of the conventional and new expression of h and their influence on the accuracy of the step response in one layer slab, RC=1.59 1/h and L=0.05 m. The figure illustrates step responses when time step η equals 0.63RC, 0.31RC and 0.06RC from up to down. Conventional h means the relationship h=η.

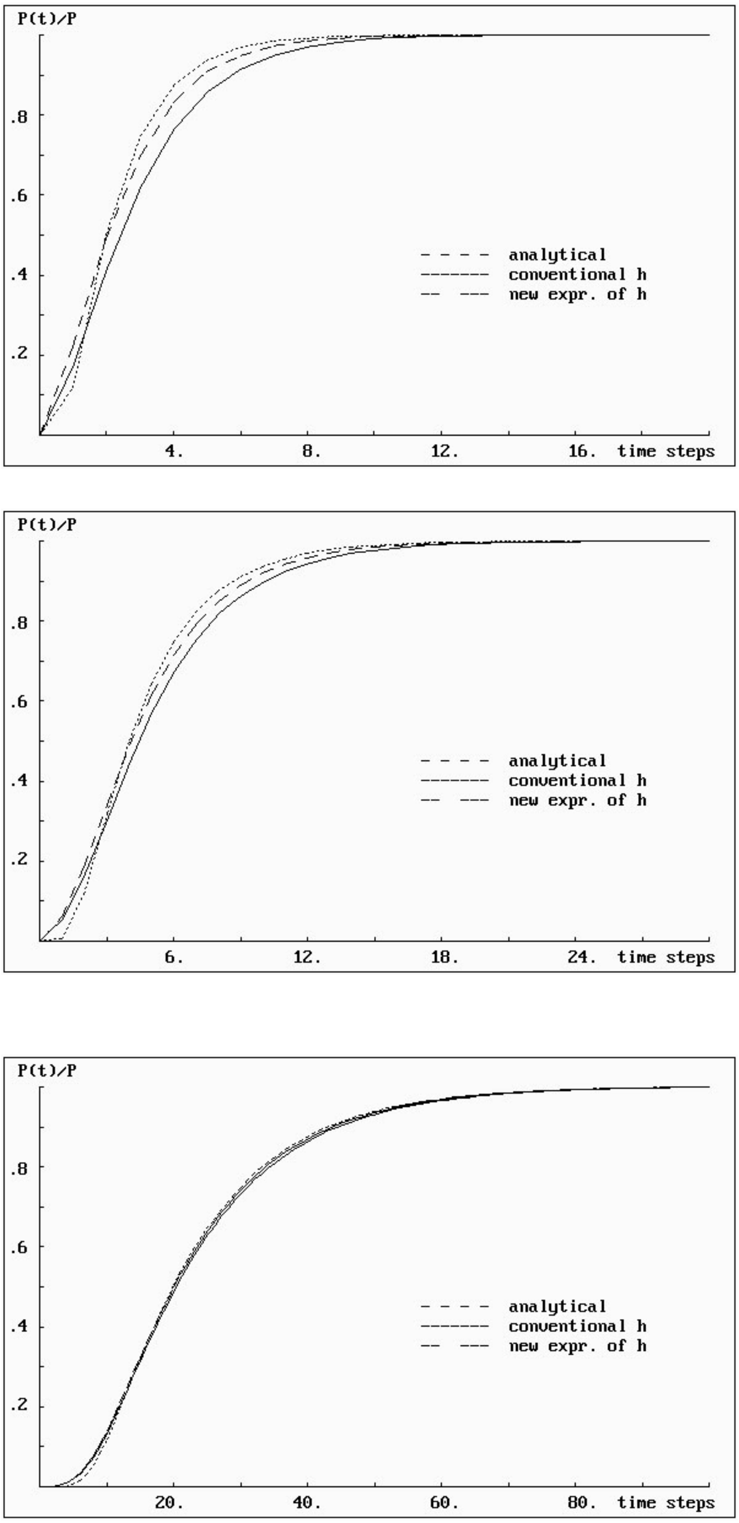


Figure 2. Comparison of the conventional and new expression of h and their influence on the accuracy of the step response in one layer slab, RC= 14.3 1/h and L=0.15 m. The figure illustrates step responses when time step η equals 0.070RC, 0.030RC and 0.007RC from up to down. Conventional h means the relationship h=η.

7 Performance and comparision of the method

7.1 The differential-difference method compared with the explicit numerical method

The explicit method

Application and development of numerical methods to the problems of heat conduction has created a number of techniques. One of them is the classical explicit method. This method is chosen as a representative of the numerical methods and compared with the differential-difference method.

The algorithm for calculation of the approximate temperature $\hat{u}$ inside a solid is derived starting from a differential equation of heat conduction with discretized space and time variable. The resulting algorithm is

$$\hat{u}_i^{n+1} = M(\hat{u}_{i+1} + \hat{u}_{i-1})^n - (2M - 1)\hat{u}_i^n \tag{93}$$

where subscripts and superscripts refer to nodal points of x and t. The modulus M is defined as

$$M = \frac{\Delta t}{R_i C_i} \tag{94}$$

R_iC_i in this equation is a time constant of one discrete part of the slab, thickness of Δx, located at node i. The explicit method is simple to use, but one must choose the right time and space subintervals in order to prevent instabilities. In order to maintain the stability, it is required that $M \leq 0.5$.

One layer slab

The methods are compared by using several responses and time steps. Results, computed using analytical methods, are applied as reference values. Figures 3 to 9 present test runs of step and ramp responses and deviations from exact solutions. Figures 5, 7, 9, and 11 illustrate heat flow rates at the surface as a response to the temperature change on the same or opposite surface of the slab. The other figures present absolute percentage errors of both methods compared with the exact solution. Observe that the exact solutions are also presented using a curve connecting points at discrete intervals. The practice is the same for all the figures. Only errors up to 100 % are shown.

The following conclusions were made after the comparison. First, accuracy of the differential-difference method is better in most cases. Especially figures 3, 4, 9 and 10 show a good performance of the differential-difference method. Also in figures 5 and 6 the new method indicates better performance. Only when the time step is small compared with the time constant *RC*, the explicit method seems to be better. In this region differences in time derivatives are negligible and the number of nodal points inside slab is high enough to compensate benefits of the continuous space variable of the differential-difference method.

The responses of figures 7 and 8 represent the only case where the explicit method shows a better performance. The differential-difference method is not able to process correctly the high frequencies of the response. This is due to the lumping constant *h*, which was not chosen for small times of the response. When the frequencies are lower, the differential-different method shows a good performance as indicated in figures 9 and 10.

Comparison of the methods can be performed only in a relatively narrow time step region because of unstable behaviour of the explicit method. It needs separate control and adjustments for every layer and the same time step cannot be applied to all slab thicknesses. The differential-difference method has no such problems, and it is inherently stable for all time steps, except some extreme regions.

Table 5 illustrates some results of the comparison. It presents percentage errors of both methods with different time steps and the number of coefficient functions and nodes in each case. The number of coefficient functions is two or three times greater than the number of nodes in explicit method. If more coefficient functions are included they do not improve the accuracy. The number of coefficients is prescribed using the method of chapter 3.4.

Computing time

In order to examine the computing time of both methods in similar circumstances, the solution concerning the one-layer slab was converted to a computer program. Both programs computed the heat flow rate at the surface of the slab, when the temperature of the opposite surface changed stepwise. The initial temperature of the slab was zero. Equations (93) and (94) were modified and applied to the explicit method and Equation (65) to the differential-difference method. The coefficient functions were calculated using the procedure of Table 1. Appendix 2 presents the computer programs of both methods. Test runs were performed for several time steps between 0.006*RC* and 0.06*RC*. They show that the differential-difference method needs 7 to 10 % more computing time in each case.

Table 2. Comparison of the differential-difference and explicit method in a one-layer environment. Properties of the slab are: L = 0.375 m, RC = 26.35 1/h.

Time step	**Differential-difference**		**Explicit**	
η/RC	**error [%]**	**coefficients**	**error [%]**	**nodes**
0.006	0.11	17	0.12	8
0.01	0.16	13	0.16	6
0.02	0.31	9	0.37	4
0.04	0.52	6	0.51	2
0.06	0.63	5	0.73	1

Note that the computer programs include about the same number of program lines. The explicit numerical method is known as one of the simplest procedures for numerical heat transfer. The compact form of the differential-difference computer program is due to the recursive property of the method. Both the coefficient functions and the solution are calculated with simple recursive formulas.

A multi-layer slab or wall

Although the computing time of the explicit method is slightly shorter in one layer slab, the situation is quite different in multi-layer environment. While applying the explicit method, the time increment and spatial nodal points must be set according to the thickness and material properties of the solid. Especially walls may contain several layers, many of them thin chipboards, plasterboards or coatings, which cannot be approximated as purely resistive layers. Usually the chosen time step must be relatively short otherwise the algorithm may become unstable. This feature is clearly seen even in a wall of considerably thick and massive layers. Table 3 presents a similar comparison as Table 2 The differential-difference method has no difficulties to compute the heat flow rate, but the explicit method can be applied only if the shortest time step is chosen. Even in this case the accuracy of differential-difference method is better.

Table 3. Comparison of the differential-difference and explicit numerical method in a multi-layer environment. Four layers and their properties are: concrete L = 0.05 m, RC = 1.59 1/h, air gap R = 0.17 m^2K/W, insulation L = 0.175 m, RC = 5.23 1/h and concrete L = 0.05 m, RC = 1.59 1/h.

Time step	**Differential-difference**		**Explicit**	
η/RC	**error [%]**	**coefficients**	**error [%]**	**nodes**
0.006	0.05	14	0.32	6
0.01	0.06	12	-	-
0.02	0.11	9	-	-
0.04	0.15	7	-	-
0.06	0.20	6	-	-

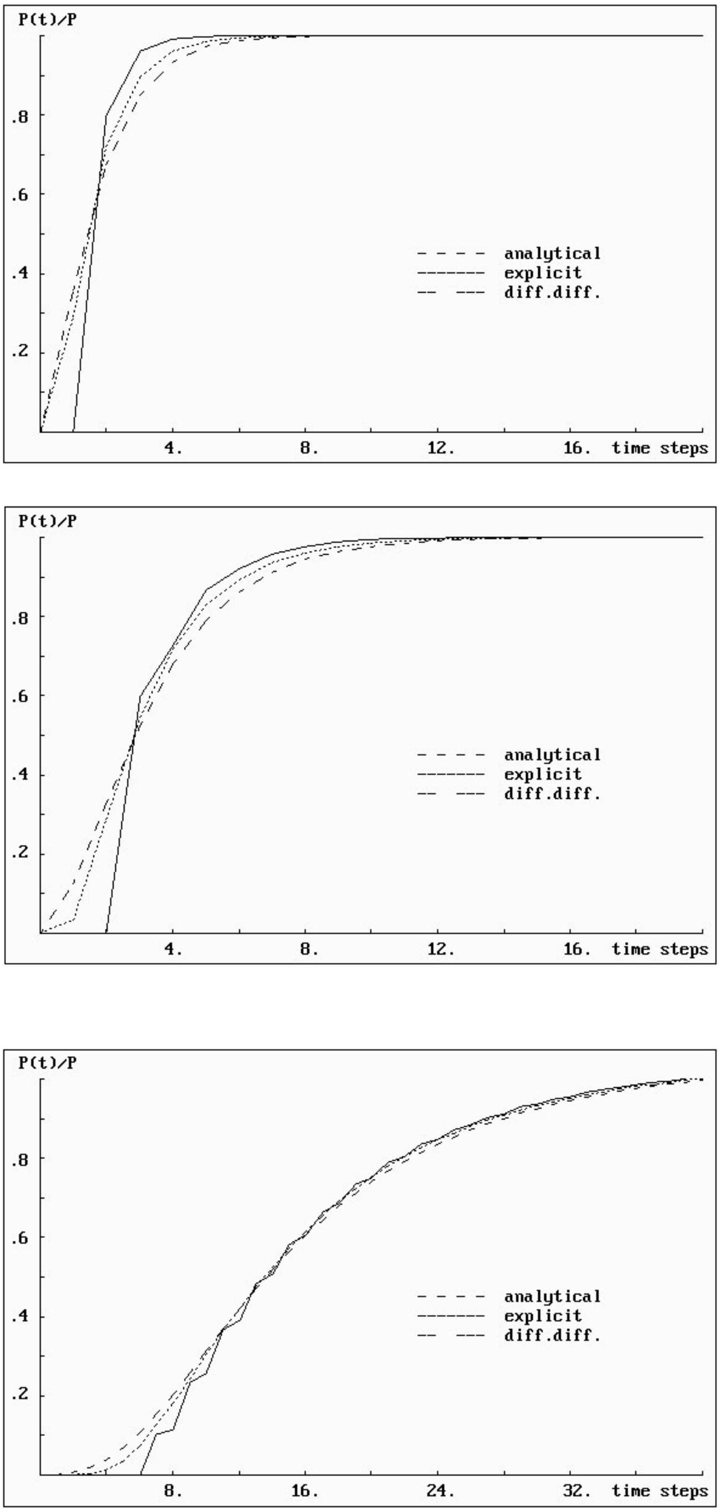


Figure 3. Comparison of the explicit and differential-difference methods in a one-layer slab where L = 0.15 m and RC = 14.3 1/h. The figure illustrates heat flow rate in the plane x = 0, when the temperature at x = L changes stepwise. The applied time steps are 0.1RC, 0.05RC and 0.01RC from up to down.

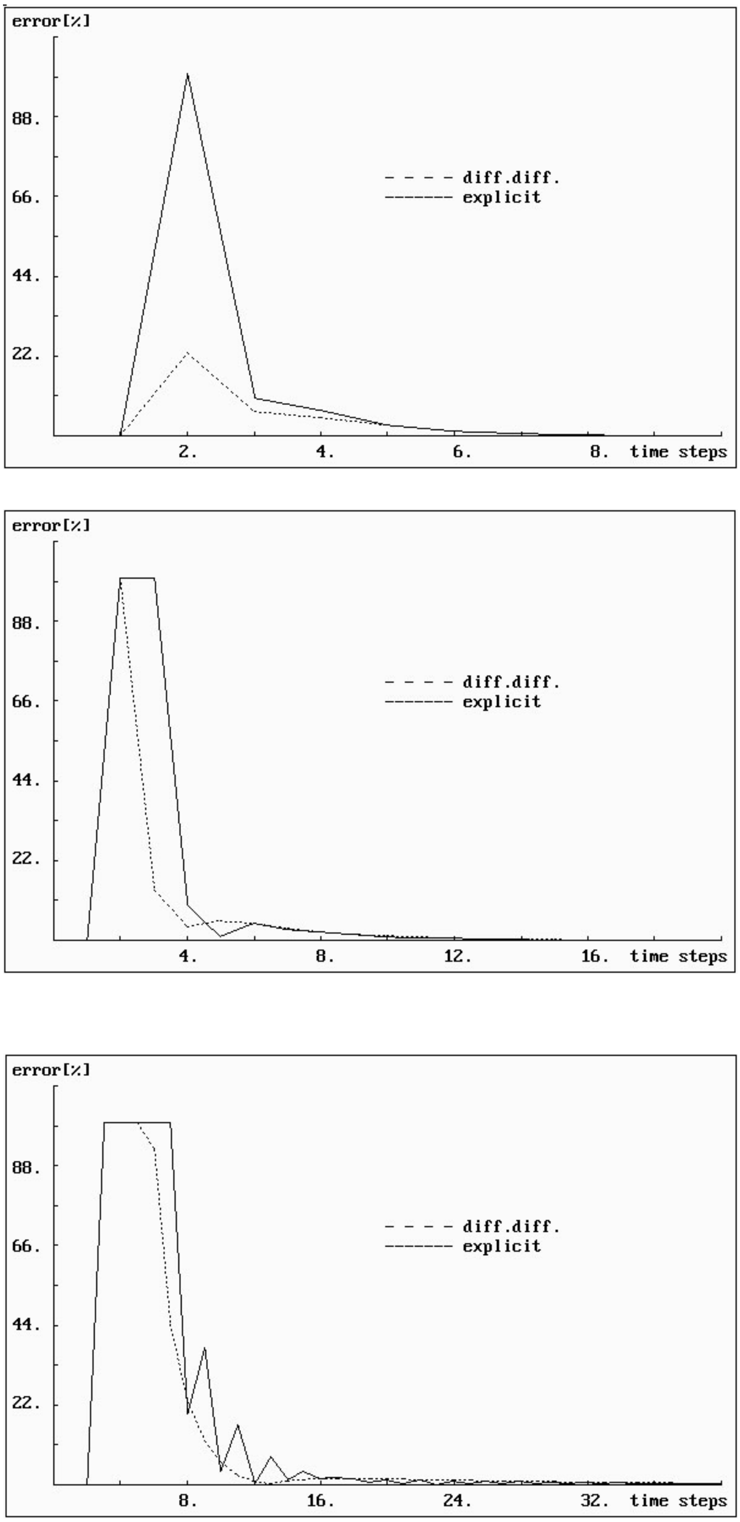


Figure 4. Comparison of the explicit and differential-difference methods in a one-layer slab where L = 0.15 m and RC = 14.3 1/h. The figure illustrates percentage errors in each case of figure 3.

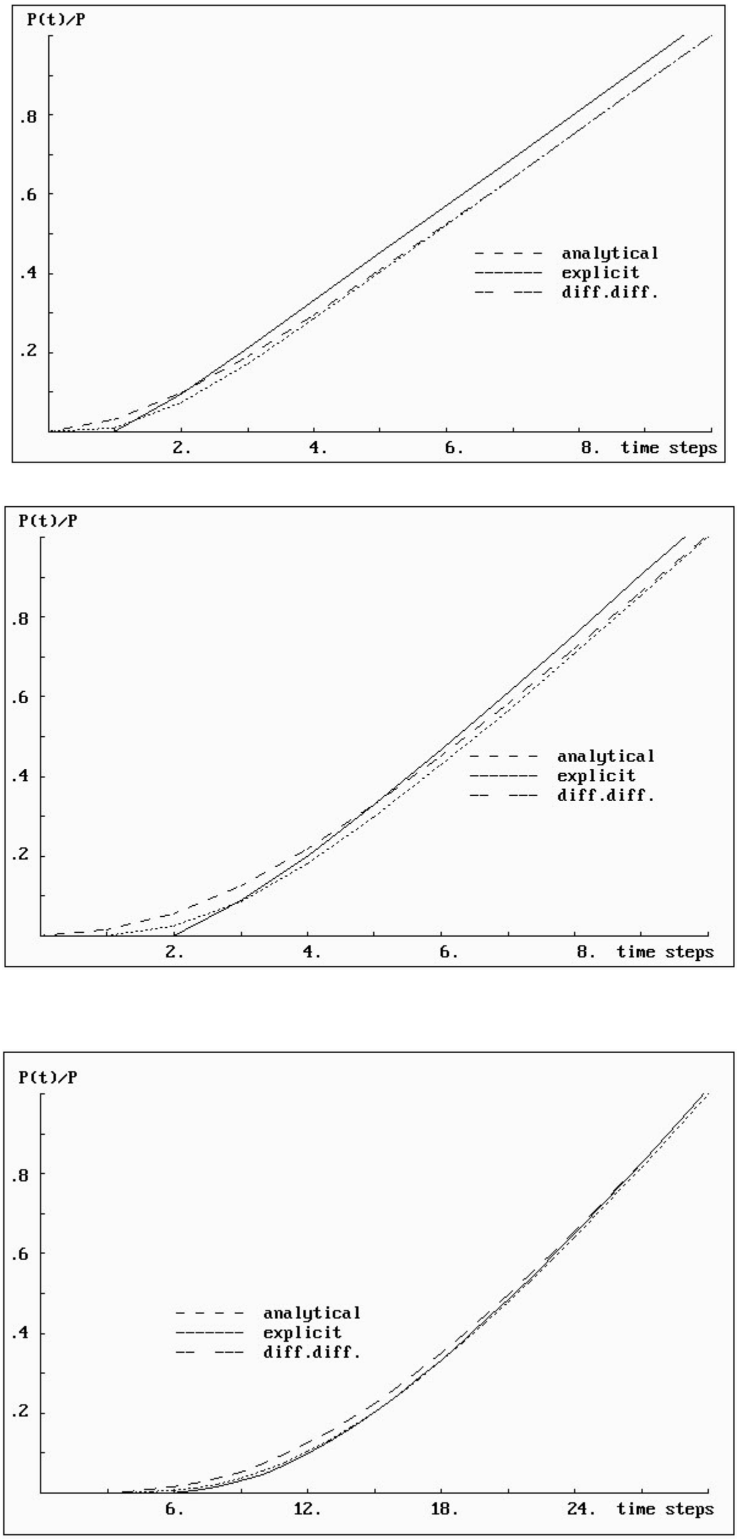


Figure 5. Comparison of explicit and differential-difference methods in a one-layer slab where L = 0.15 m and RC = 14.3 1/h. The figure illustrates heat flow rate in the plane x = 0, when the temperature at x = L changes rampwise. The time steps are 0.1RC, 0.05RC and 0.01RC from up to down.

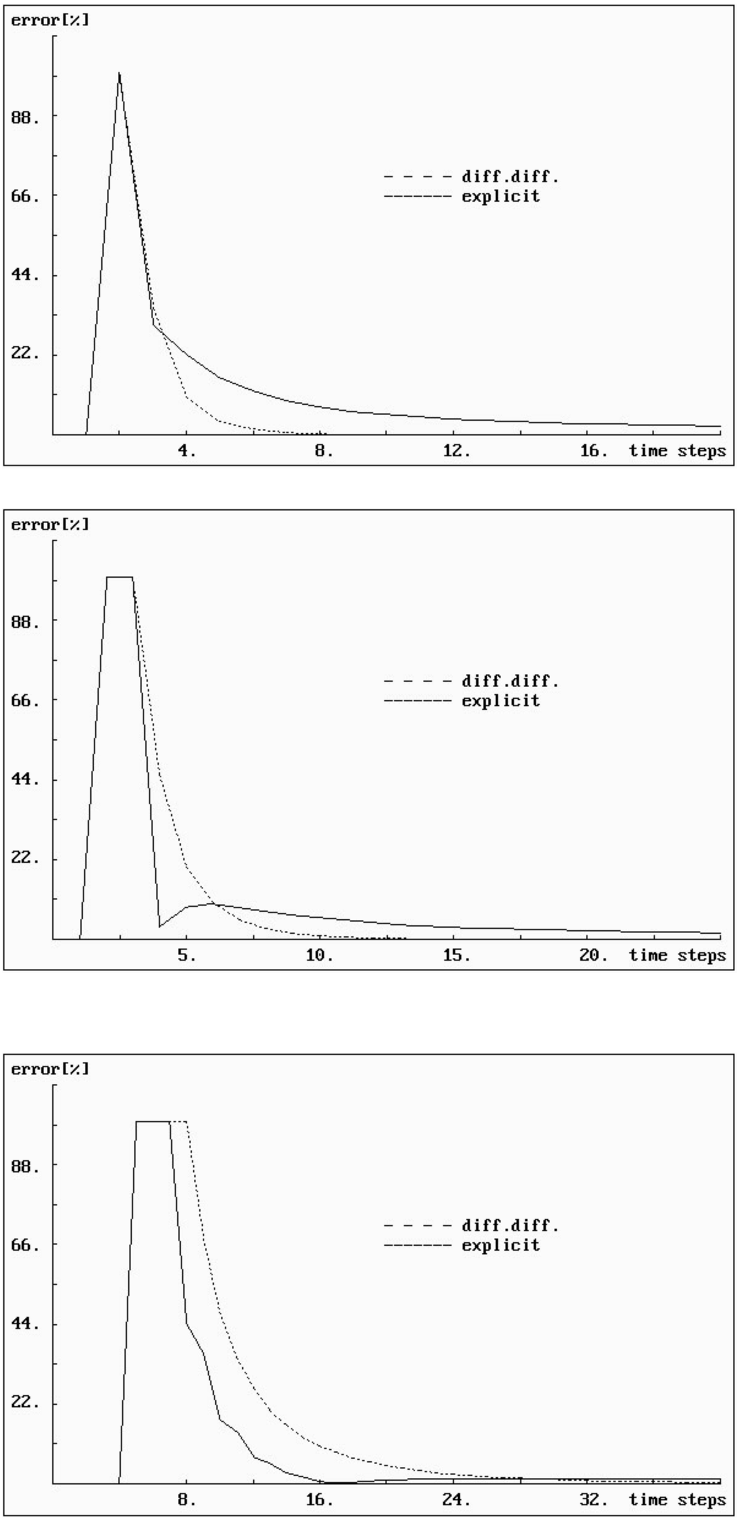


Figure 6. Comparison of the explicit and differential-difference methods in a one layer slab where L = 0.15 m and RC = 14.3 1/h. The figure illustrates percentage errors in each case of figure 5.

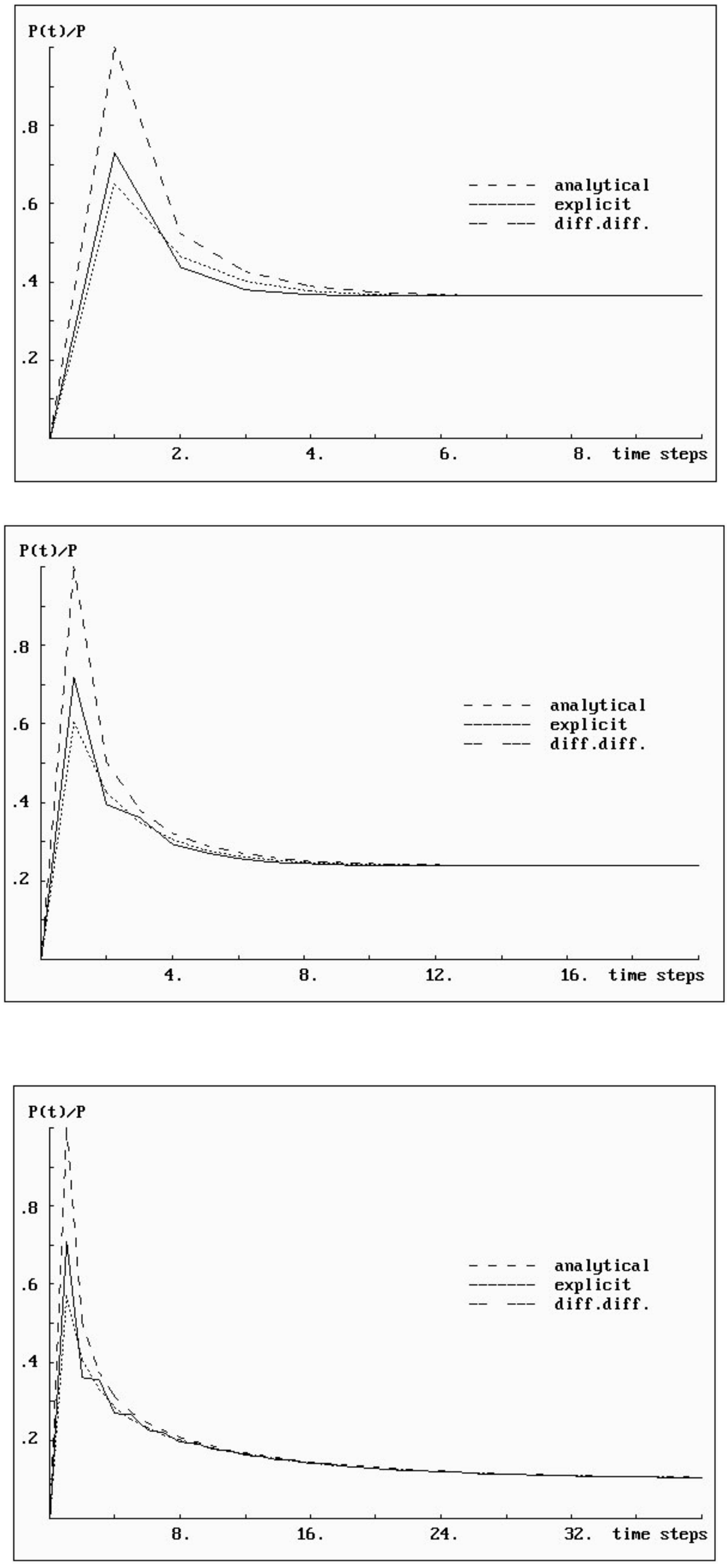
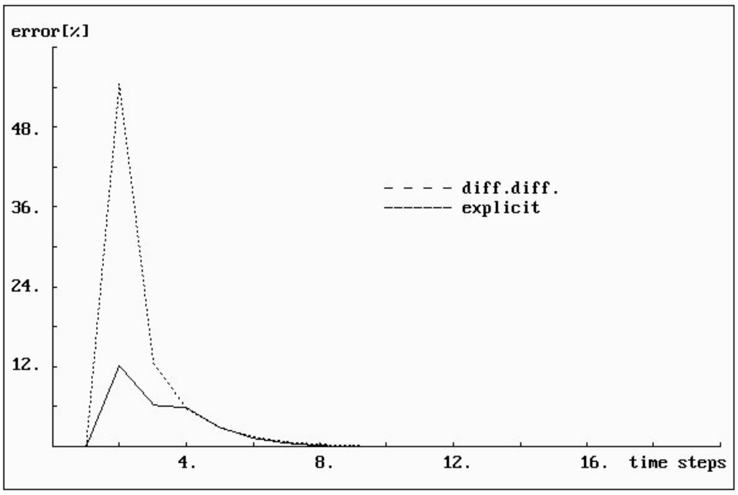
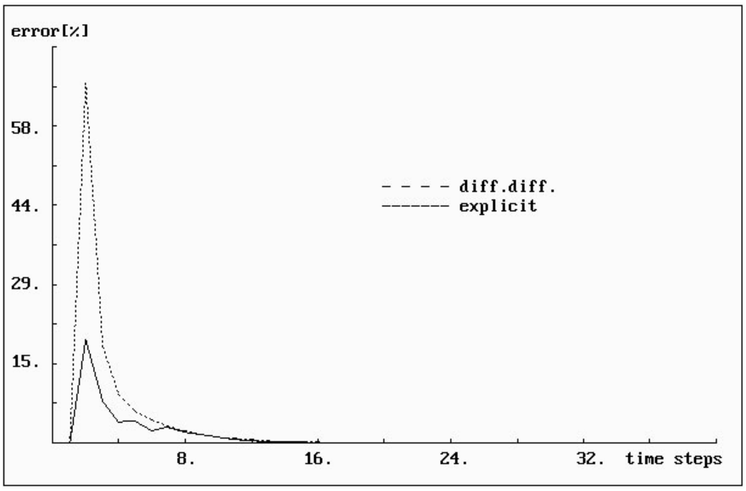


Figure 7. Comparison of the explicit and differential-difference methods in a one layer slab where L = 0.15 m and RC = 14.3 1/h. The figure illustrates heat flow rate in the plane x = 0, when the temperature at x = 0 changes stepwise. The time steps are 0.1RC, 0.05RC and 0.01RC from up to down.





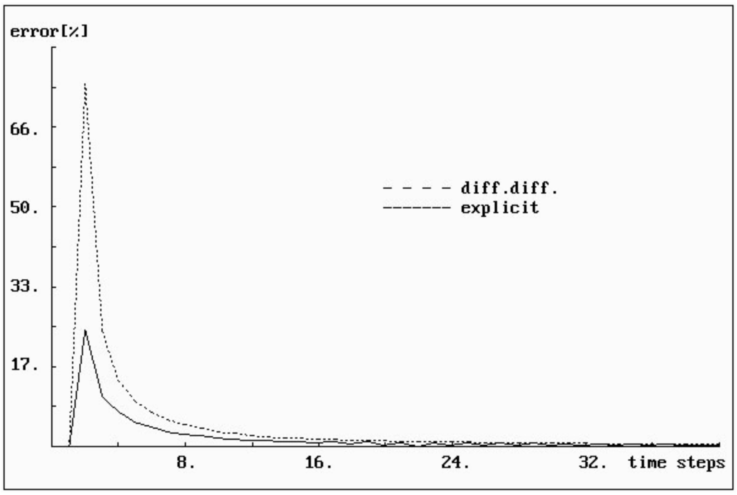


Figure 8. Comparison of the explicit and differential-difference methods in a one-layer slab where L = 0.15 m and RC = 14.3 1/h. The figure illustrates percentage errors in each case of figure 7.

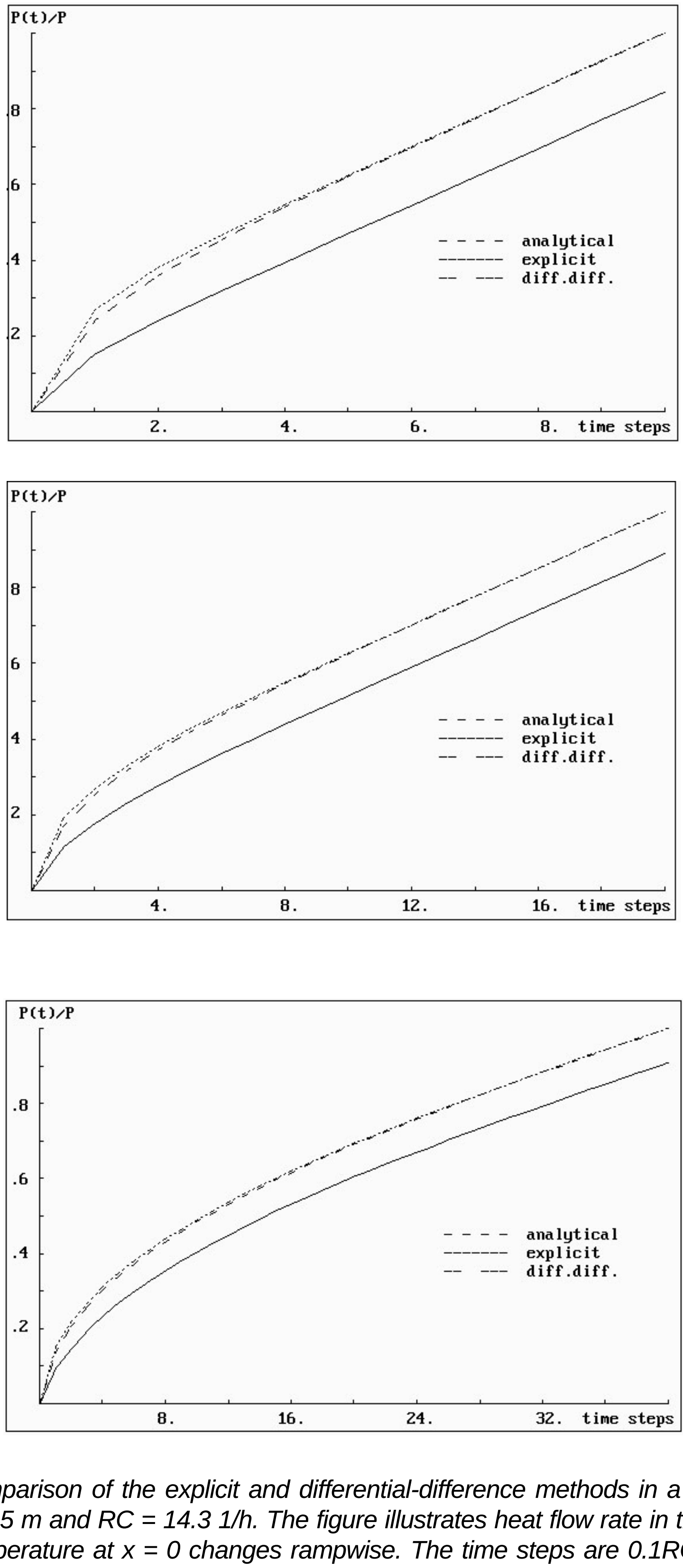
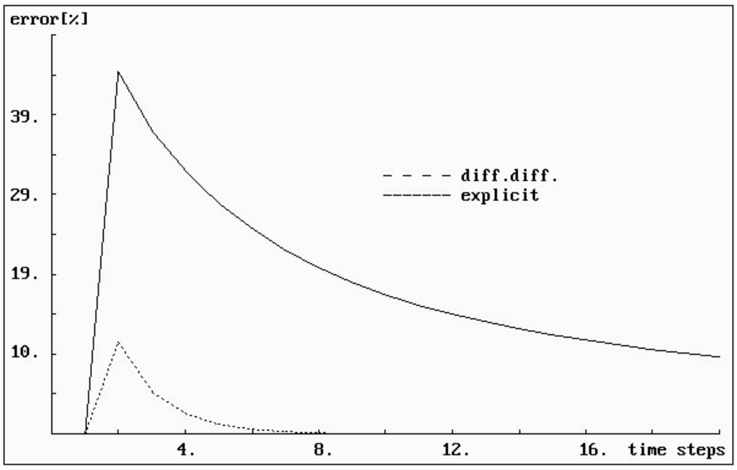
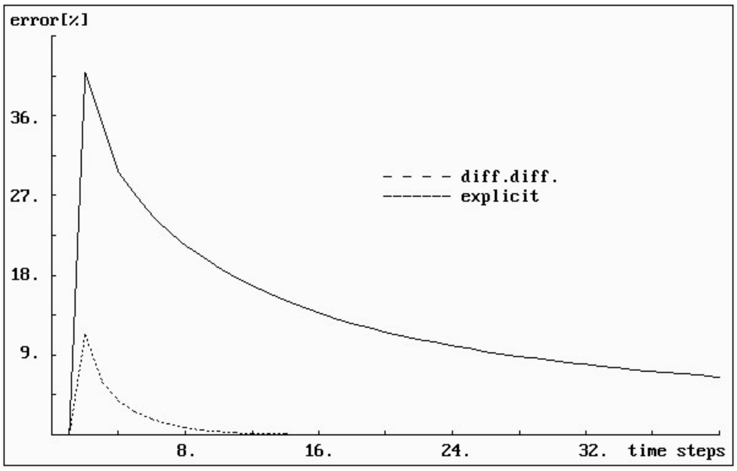


Figure 9. Comparison of the explicit and differential-difference methods in a one-layer slab where, L = 0.15 m and RC = 14.3 1/h. The figure illustrates heat flow rate in the plane x = 0, when the temperature at x = 0 changes rampwise. The time steps are 0.1RC, 0.05RC and 0.01RC from up to down.





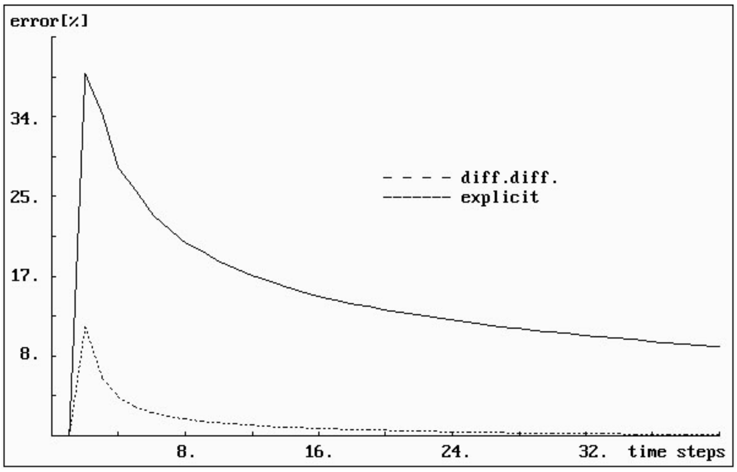


Figure 10. Comparison of the explicit and differential-difference method in a one-layer slab where L = 0.15 m and RC = 14.3 1/h. The figure illustrates percentage errors for each case of figure 9.

7.2 The differential-difference method compared with the pulse response method

The pulse response method is a conventional approach when heat flow through a multi-layer slab or wall is computed. The method has been developed from analytical solution of heat conduction. Inputs are overlapped triangular pulses forming an approximate variation of any sampled input signal. Calculation of the response factors utilizes root finding algorithms, and matrix calculus.

Formally, the equations describing heat flow rate by means of the pulse response method and the differential-difference method have several common features. The differences between these two methods are clearly seen from the following set of equations. According to Park et. al (1986) the heat flow rates at the surfaces of a one-layer slab using the pulse response method are the following

$$P_1(t) = \sum_{k=1}^{K} R_k P_1(t-k) + \sum_{m=0}^{M} Y_k U_2(t-m) + \sum_{m=0}^{M} Z_m U_1(t-m) \qquad (95)$$

$$P_2(t) = \sum_{k=1}^{K} R_k P_2(t-k) + \sum_{m=0}^{M} Y_k U_1(t-m) + \sum_{m=0}^{M} X_m U_2(t-m) \ , \qquad (96)$$

where X_m, Y_m and Z_m are called the external, cross and internal transfer functions. R_k refers to past heat flow rates. When the heat flow rates produced by the differential-difference method are modified to the same format as above, they can be written as:

$$P_1(t) = -\frac{1}{b_0}\sum_{n=1}^{N} b_n P_1(t-n) - \frac{kq}{b_0}U_2(t) + \frac{kq}{b_0}\sum_{j=1}^{J} a_j U_1(t-j) \qquad (97)$$

$$P_2(t) = -\frac{1}{b_0}\sum_{n=1}^{N} b_n P_2(t-n) - \frac{kq}{b_0}U_1(t) + \frac{kq}{b_0}\sum_{j=1}^{J} b'_j U_1(t-j) \qquad (98)$$

A notable difference between the two methods is the number of the cross transfer functions Y_m, m = 0, 1, 2.... The latter equations contain only one cross transfer function at m = 0. Other parts of the equations are formally similar in both cases. When methods are applied to the same slab with equal time steps, test runs show that $N \geq K$ and $J \geq M$, i.e. some variables of the differential-difference method need more transfer functions but the total number of them is about the same in both methods.

Clearly, the pulse response method generates more accurate results and one also has a possibility to control the accuracy. In principle, one needs only to increase the number of pulse response factors to achieve numerical results closer to their exact values. The differential-difference method is able to produce approximate results, and better accuracy is achieved only by decreasing the time step. Figures 11, 12, 13, and 14 illustrate the ability of the differential-difference method to represent a step and ramp response in a heavy and light multi-layer wall. The corresponding pulse response factors according to (97) and (98) are presented in appendix 3.

Walton (1984) has studied the pulse response method. He presents the following Table 4 which describes the ability of the pulse response method to compute heat flow through thin and massive layers. The table also shows the number of *M* and *K* in each case. The pulse response method has difficulties in two areas which are indicated by hyphens and stars. The hyphens represent an area of long time steps and thin layers. Errors in the dynamic response of the slab are noticeable when the number of transfer functions is less than three. In the area of stars the number of transfer functions and their digits increase. This means problems due to the round-off errors. Walton used 32 bit real numbers.

When the similar test runs are applied to the differential-difference method, conclusions are the same as above in the area of stars (Table 5. The number of coefficient functions increase beyond twenty. The floating point math package, built inside the computer program had problems to handle the large numbers. In the area of hyphens calculations were carried out without any difficulties. The coefficient functions are close to the same magnitude which is their steady state value.

Table 4. Number of transfer functions (M/K) required for modelling a homogeneous slab computed to 0.001 percent accuracy in a steady state value presented by Walton (1984). Time steps are in seconds and slab thicknesses in meters, diffusivity κ=0.0029 h/m^2.

Time	**Thickness Of The Slab**							
Step	**.013**	**.025**	**.037**	**.051**	**.076**	**.102**	**.152**	**.305**
3600	-	-	2/1	2/1	3/1	3/1	4/1	6/3
900	-	2/1	3/1	3/1	4/1	4/2	6/3	10/5
240	2/1	3/1	4/1	4/2	5/3	7/3	9/5	*
60	3/1	4/2	5/3	7/3	9/5	13/5	*	*
15	4/2	7/3	9/5	13/5	*	*	*	*
5	6/3	11/5	*	*	*	*	*	*

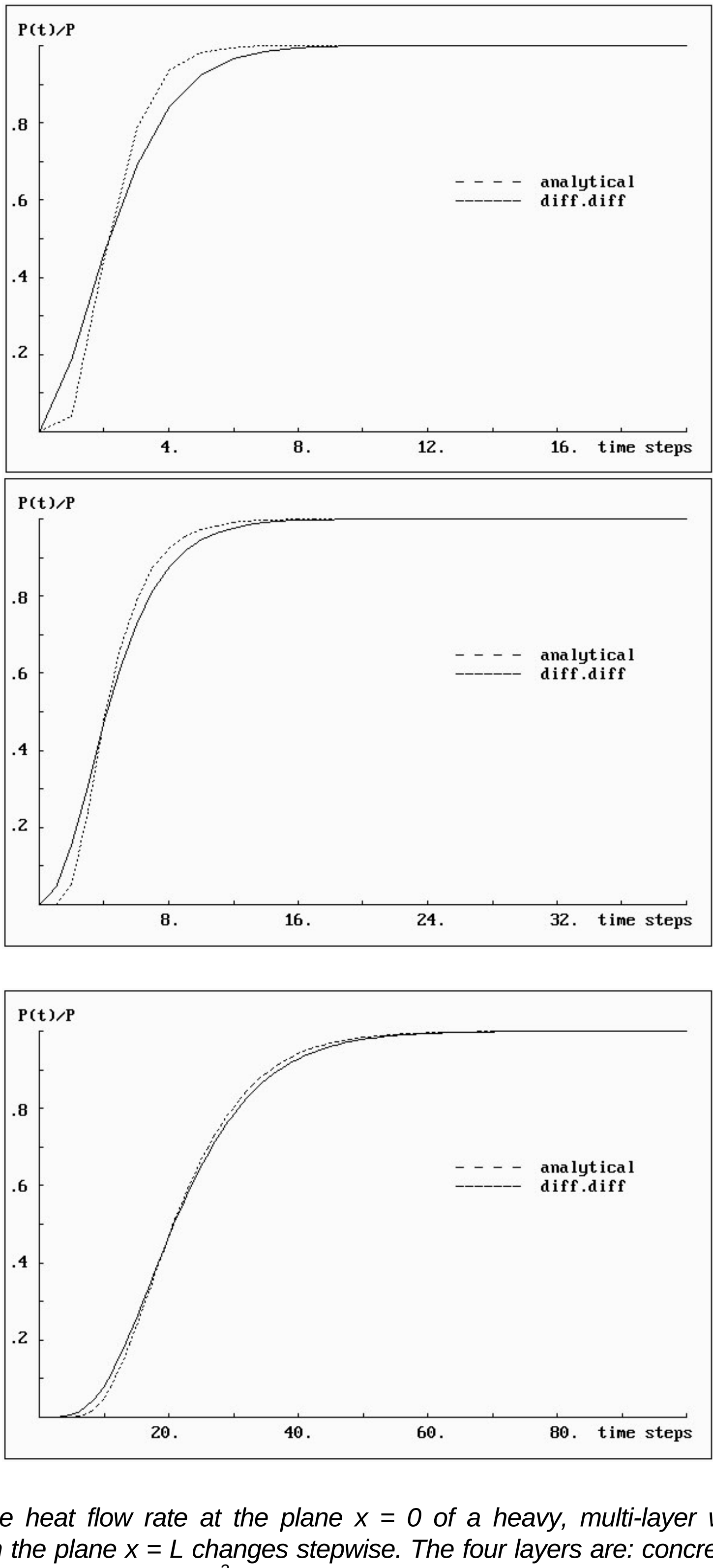


Figure 11. The heat flow rate at the plane x = 0 of a heavy, multi-layer wall, when the temperature in the plane x = L changes stepwise. The four layers are: concrete L = 0.05 m, RC = 1.59 1/h, air gap R = 0.17 m^2K/W, insulation L = 0.175 m, RC = 5.23 1/h and concrete L = 0.05 m, RC=1.59 1/h. Time steps are 0.050RC_{ALL}, 0.025RC_{ALL} and 0.005RC_{ALL} from up to down.

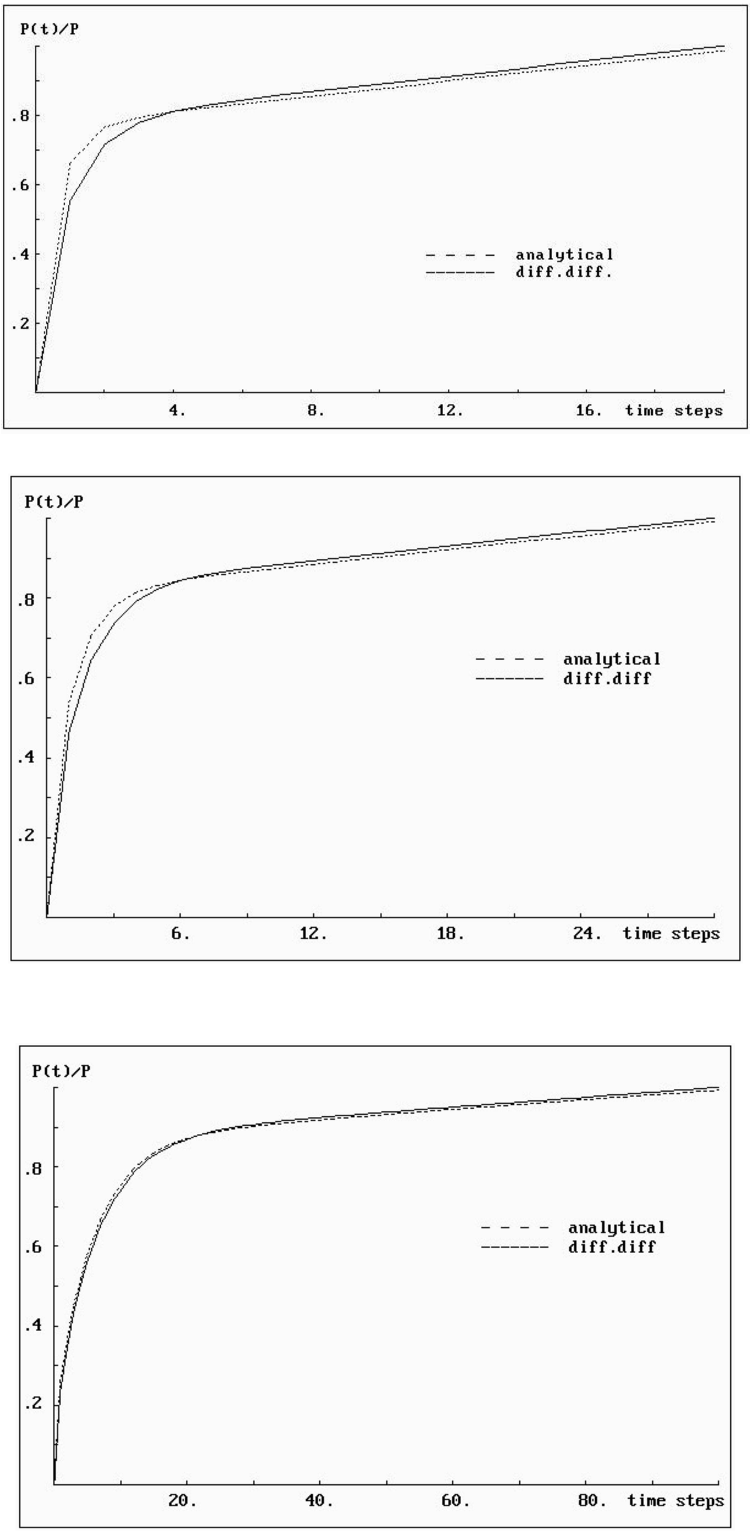


Figure 12. The heat flow rate at the plane x = 0 of a heavy, multi-layer wall, when the temperature in the plane x = L changes rampwise. The four layers are: concrete L = 0.05 m, RC = 1.59 1/h, air gap R = 0.17 m^2K/W, insulation L = 0.175 m, RC = 5.23 1/h and concrete L = 0.05 m, RC = 1.59 1/h. The time steps are are 0.050RC_{ALL}, 0.025RC_{ALL} and 0.005RC_{ALL} from up to down.

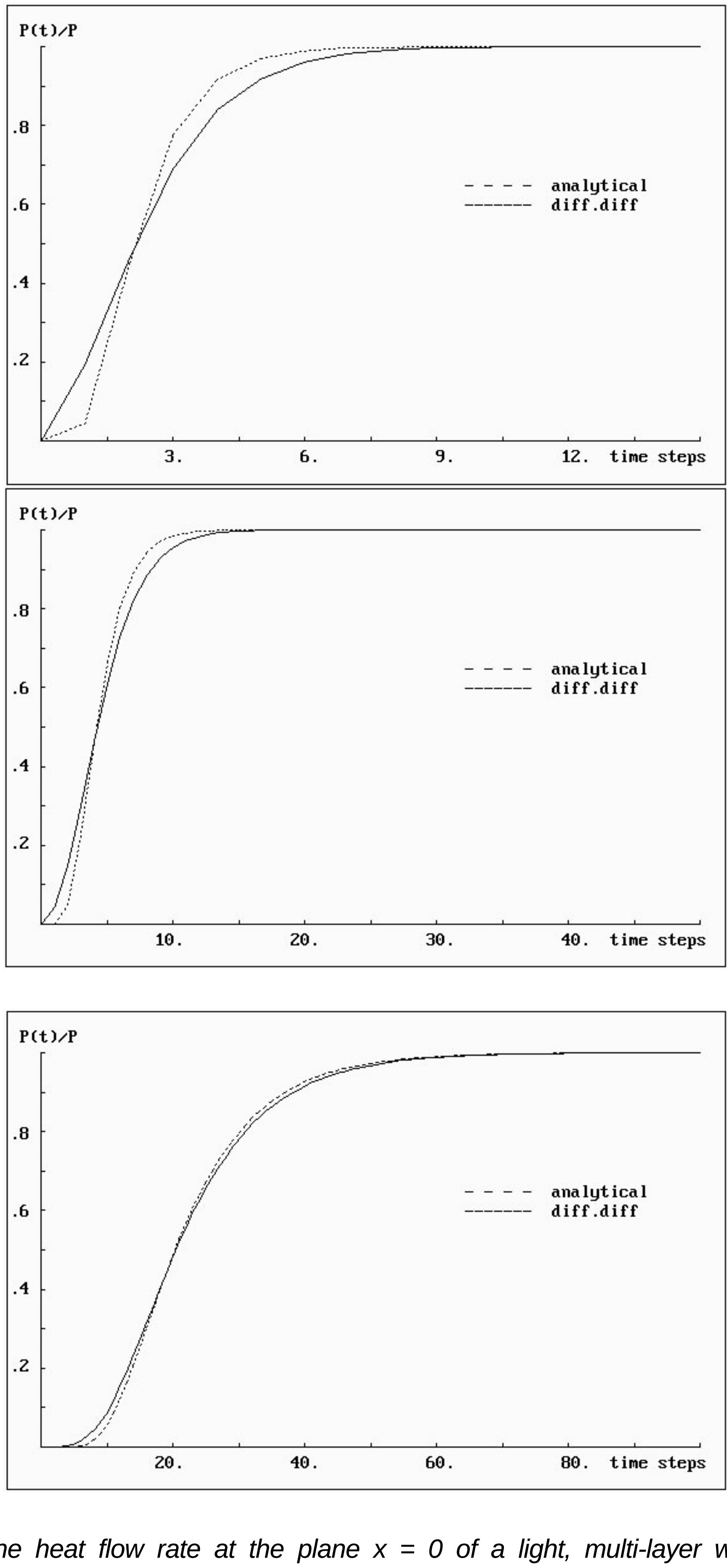


Figure 13. The heat flow rate at the plane x = 0 of a light, multi-layer wall, when the temperature in the plane x = L changes stepwise The four layers are: wood L = 0.02 m, RC = 0.50 1/h, air gap R = 0.17 m^2K/W, insulator L = 0.175 m, RC = 5.23 1/h and wood L = 0.02 m, RC = 0.50 1/h. The time steps are $0.050RC_{ALL}$, $0.025RC_{ALL}$ and $0.005RC_{ALL}$ from up to down.

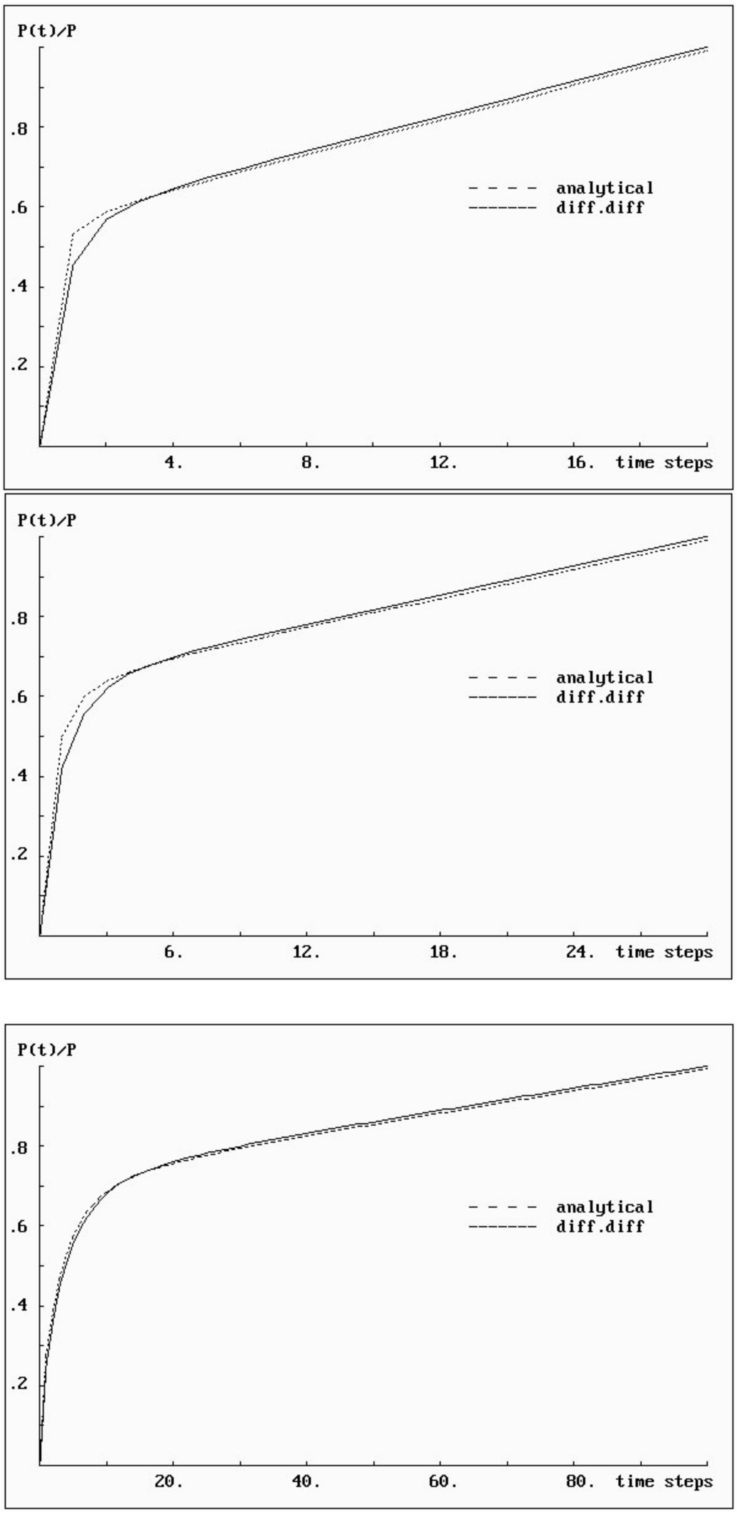


Figure 14. The heat flow rate at the plane x = 0 of a light, multi-layer wall, when the temperature in the plane x = L changes rampwise. The four layers are: wood L = 0.02 m, RC = 0.50 1/h, air gap R = 0.17 m^2K/W, insulator L = 0.175 m, RC = 5.23 1/h and wood L = 0.02 m, RC = 0.50 1/h. The time steps are 0.050RC_{ALL}, 0.025RC_{ALL} and 0.005RC_{ALL} from up to down.

7.3 Time step and slab thickness vs. accuracy

Due to the time derivative approximation, numerical errors of the differential-difference approach are proportional to the selected time step. Smaller time steps usually means smaller absolute errors. The same feature is common to all numerical heat transfer methods. Consider a test case where relative errors are computed by summing at every time instant the difference between the analytical and numerical results. Then these sums are divided by the number of time steps and the steady state value of the response. In this way relative errors, characteristic of all responses are found. Table 5 shows these results as a function of time step size and thickness of the slab. These parameters are the same as in Table 4. Zeros in the upper left corner are due to the single coefficient function which is almost equal to the corresponding analytic value. The chosen time step and thickness of the slab have a noticeable effect on the relative error. Decreasing time step and increasing thickness makes the relative error smaller, until round-off errors start to cumulate. Distribution of the errors is also effected by the value of the lumping constant *h*. The table clearly shows combinations of those time steps and thicknesses which are not most appropriate for the chosen value of *h*.

Table 5. Relative errors of the differential-difference method as a function of time step and thickness. Time step sizes are in seconds and thicknesses in meters. Material properties are the same as in Table 4.

Time	**Thickness Of The Slab**							
Step	**.013**	**.025**	**.037**	**.051**	**.076**	**.102**	**.152**	**.305**
3600	0.00	0.00	0.00	0.00	0.28	1.05	1.45	1.03
900	0.00	0.04	0.80	1.11	1.37	1.12	0.65	0.24
240	0.01	0.85	1.05	1.31	0.97	1.00	0.31	*
60	0.83	1.43	0.96	1.00	0.29	0.17	*	*
15	1.33	0.83	0.29	0.16	*	*	*	*
5	0.67	0.23	*	*	*	*	*	*

7.4 Errors involved in the differential-difference scheme

Numerical solutions are subject to several types of errors. Numerical calculations are carried out using a finite number of decimal places. Rounding a decimal number at each step causes a small error. Although the error is small in size, their cumulative effect may be significant.

Many numerical solutions are based on finite difference approximation of the time and space variable. Finite differences are derived from truncated Taylor series expansion. Truncation of the series causes an error at each step of the solution. Decreasing of the time and space subdivisions also decreases the error. If the mesh size becomes small, computing takes a longer time. Thus, the mesh size is a compromise.

Similar errors also concern the differential-difference approach. However, the test runs described in chapter 6 and the results of Table 5 indicate that numerical errors are not strictly linearly dependent of the time step size. For each response there probably exists an optimum step size which also has a slight decreasing effect on the numerical error. When several layers are combined the situation becomes more complex.

8 Applicability of the method to dynamic thermal models

8.1 Dynamic thermal models of an occupied space

A dynamic thermal model of a building is usually based on the heat balance of an occupied space or zone. Typically, convective gains from the room surfaces are added and the sum is set equal to zero. This also includes the effect of ventilation and heat produced by internal sources. A simplified form of the heat balance of a zone can be expressed mathematically as

$$C_R \frac{dU_R}{dt} = \sum_i h_{Si} A_i \left[U_i(t) - U_R(t) \right] + \dot{C}_f U_o(t) + \dot{C}_e U_R(t) + \varphi \qquad (99)$$

where φ refers to convective loads of machines, lighting, occupation and the heating system. Variables $U_R(t)$, $U_i(t)$ and $U_o(t)$ refer to the room, surface and outdoor temperatures.

A key issue in the thermal model is how the heat balance on the surfaces of the room is computed. Characteristic features of the model will be determined by the applied algorithm. Typical is a finite difference or an analytical model, where the latter is usually based on the pulse transfer method. The differential-difference method becomes a new alternative, possessing some benefits over the old types. All these models still require that the heat balance of the zone is solved at discrete time instants.

The thermal model of the room can also be formulated with purely analytical expressions, which means evaluating all heat balances analytically. Such a solution is successful only under restricted assumptions. A semi-infinite slab is the only slab type, where a closed form solution has been presented (Rotem et al. 1963, Saastamoinen & Nylund 1980). For one and multi-layer slabs and walls such a solution involves extremely heavy algebra. Thus, a pure analytical approach is not a good choice.

8.2 Single-path models

Difficulties of the analytical approach have led to reduced, lumped parameter models. One of them is a single-path model, which represents a thermal response of the room temperature to a change in the outdoor temperature or the heating power. Only the main components of the relationship are included. Because the effect of surfaces and walls dominate the response, the physical properties of the structures must be part of the model. The response of the room temperature to the outdoor disturbances is characterized as a low frequency response whereas that due to inside disturbances is called high frequency response (Cowan 1976).

The differential-difference approach is well suited for reduced, single-path models. This kind of model is not feasible for the finite difference technique due to the inner nodal points of the slab. Also, the pulse transfer method is too complicated for such a simple model.

Single-path models have been utilized especially in small microprocessor systems. Typical implementation is a self-tuning optimum start time control of heating plant (Dexter 1981, Florez & Barney 1987), control of stored-energy heating systems (Dexter & Hayes 1981) or heating load prediction (Pakanen 1992). In these systems the model is an integral part of the control sequence. The discrete time system combined with the recursive structure of the differential-difference solution is well suited for these applications.

8.3 Larger models

If more details of the building and/or zone are added to the thermal model, obviously, the results will be better but the complexity of the system increases. Equations of such a model are feasible to process in matrix form. A small computer program for simulation of a building thermal response may contain a matrix equation consisting of ten to fifteen separate equations. The whole building is then modelled as one room a zone. A large simulation program, consisting of several hundred subroutines, computes, in addition to a detailed building envelope, the effect of internal gains, solar gains and shadowing, air infiltration, air movement between rooms, and controls of the HVAC-systems. The total building model is combined of single room or zone models.

The size and implementation of the simulation programs and their thermal models depends on their designed application areas. An energy management and control system may include a thermal simulation program for optimal energy and control strategies of buildings. It provides a convenient tool for plant operators and energy managers for diagnostics checks of energy consumption. A stand alone tool for a design engineer or architect could be a computer program with a thermal model running in a PC. The program simulates and checks the internal temperature levels of a building under different loading conditions, structures and material alternatives. An energy analysis computer program containing 25000 program lines or more represents a large thermal simulation capability of a building with a sophisticated dynamic thermal model (Walton 1983b).

The differential-difference method is suitable for all those applications. As previously shown, the differential-difference method has some benefits over numerical explicit method, such as good stability, accuracy, and performance in multi-layer environment. Thus, the numerical explicit method and probably many other numerical methods could be replaced by the differential-difference method.

When large energy analysis programs are concerned the situation is different. Such programs typically use a long time step and they are optimized for multi-season applications. There is no need to compute the transfer functions repeatedly, and running of the complete energy analysis may take a long time. Preparations for the run, including numerical solution of transcendental equations of multi-layer walls, takes a time which is a negligible part of the whole run. Besides the differential-difference method, there are other methods, like the pulse transfer method, which are proper for such applications.

8.4 Restrictions of the time step

One criteria for choosing proper applications for the differential-difference method is the size of time step. The test runs indicated that good accuracy is achieved if a relatively short time step is used. Although proper time step depends on several conditions, time steps less than one hour are probably best for multi-layer walls of actual buildings. If the time step is larger, decreasing accuracy cannot be avoided. Thus, an energy analysis program, where a large time step is a requirement, is not suitable according to this principle. Conversely, there are many applications where a large time step is not essential. The same is true for computer programs implemented for design, energy management, and especially for control applications.

9 Summary

The differential-difference approach for initial and boundary value problems of heat conduction find straightforward solutions in semi-infinite, one- and multi-layer slabs. All these solutions can be written with the aid of hyperbolic and exponential functions and powers. Recurrence relations of the functions make numerical algorithms short and effective. The solution can be presented either in the form of a zeroth or k^{th}-order transfer function, and the number of coefficient functions is about the same as with the pulse response method.

The differential-difference solution consists of an analytical space variable and discrete time variable. Therefore many its properties fall between numerical and analytical approaches. The accuracy and stability are better than that of an explicit numerical method but reaches the accuracy of the analytical methods only when time step is small. Numerical algorithms of the differential-difference temperature functions are slightly more sophisticated than those of basic numerical methods. Conversely, the algorithm demands no processing of transcendental equations which is typical of an analytical approach. Consequently, the differential-difference method suits for dynamic thermal models of buildings in design, energy management, and control applications.

References

Bateman, H. 1943. Some simple differential difference equations and the related functions. Bulletin of the American Mathematical Society, Vol. 49, pp. 494–512.

Bellman, R. & Cooke, K. 1963. Differential-difference equations. New York: Academic press. 462 p.

Bulavin, P. E. & Kaschcheev, V. M. 1965. Solution of nonhomogeneous heat conduction equation for multilayered bodies. International Chemical Engineering, Vol. 5, No. 1, pp. 112–115.

Butler, R. 1984. The computation of heat flows through multi-layer slabs. Building and Environment Vol. 19, No. 3, pp. 197–206.

Carslaw, H. & Jaeger, J. 1959. Conduction of heat in solids. 2nd ed. London: Oxford University Press. 510 p.

Cowan, D. 1976. The thermal response of closed environment with particular emphasis on high frequency behaviour. Dissertation. Dublin, Ireland: University of Dublin. 200 p.

Day, B. 1982. Computation of the dynamic thermal performance of buildings. Computer-aided design, Vol. 14, No. 1, pp. 49–54.

Dexter, A. L. 1981. Self-tuning optimum start time control of heating plant. Automatica, Vol. 17, No. 3, pp. 483–492.

Dexter, A.L. & Hayes, R.G. 1981. Self-tuning control scheme for domestic stored-energy heating systems. IEE Proceedings D, Vol. 128, No. 6, pp. 292–300.

Florez, J. & Barney, G.C. 1987. Adaptive control of central heating systems: Part 1: Optimum start time control. Applied Mathematical Modelling, Vol. 11, pp. 89–95.

Hittle, D. C. 1982. One dimensional heat conduction through multilayered slabs. In: Courville, E. & Bales, E. Proceedings of the building thermal mass seminar, Knoxville, TN, 2–3 June 1982. Oak Ridge National Laboratory. Pp. 405–456.

Kelly, G. E. 1988. Control system simulation in North America. Energy and Buildings, Vol. 10, pp. 193–202.

Lykov, A. V. & Mikhailov, Y. A. 1965. Theory of heat and mass transfer. Jerusalem: Israel program for scientific translations. 558 p.

May, W. B. & Park, C. 1985. Building emulator computer program for testing of energy management and control system algorithms. Washington, DC: National Bureau of Standards, Department of Commerce. 132 p. (NBSIR 85-3291)

Mehta, D. 1980. Dynamic thermal responses of buildings and systems. Dissertation. Ames, IA: Iowa State University. 241 p.

Mikhailov, M., Özisik, M. & Vulchanov, N. 1982. Diffusion in composite layers with automatic solution of the eigenvalue problem. International Journal of Heat and Mass Transfer, Vol. 26, No. 8, pp. 1131–1141.

Pakanen, J. 1992. Prediction and fault detection of building energy consumption using multi-input, single-output dynamic model. Espoo: Technical Research Centre of Finland, 32 p. + app. 12 p. (VTT Publications 116). ISBN 951-38-4234-7

Park, C., Clark, D. R. & Kelly, G. E. 1986. HVACSIM$^+$ Building systems and equipment simulation program: building load calculations. Washington, DC: National Bureau of Standards, Department of Commerce. 202 p. (NBSIR 86-3331)

Pinney, E. 1958. Ordinary difference-differential equations. Berkeley, CA: University of California press. 261 p.

Rektorys, K. 1982. The method of discretization in time. Dordrecht, Holland: D. Reidel Publishing Company. 451 p.

Roberts, A. S. & Oak, M. P. 1991. Nonlinear dynamics and control for thermal room models. ASHRAE Transactions, Vol. 97, part 1, pp. 722–728.

Rotem, Z., Gildor, J. & Solan, A. 1963. Transient heat dissipation from storage reservoirs. International Journal of Heat and Mass Transfer, Vol. 6, pp. 129–141.

Saastamoinen, J. & Nylund, C. 1980. Intermittent uppvärmning av byggnader. VST 5, pp. 15–24.

Stephenson, D. G. & Mitalas, G. P. 1971. Calculation of heat conduction transfer function for multi-layer slabs. ASHRAE Transactions, Vol. 77, pp. 117–126.

Title, C. W. 1965. Boundary-value problems in composite media: quasi-Orthogonal functions. Journal of Applied Physics, Vol. 36, 4, pp. 1486–1488.

Walton, G. N. 1983a. A computer algorithm for estimating infiltration and inter-room air flows. Washington, DC: National Bureau of Standards, Department of Commerce. 35 p. (NBSIR 83-2635)

Walton, G. N. 1983b. Thermal analysis research program reference manual (TARP). Washington, DC: National Bureau of Standards, Department of Commerce. 277 p. (NBSIR 83-2655)

Walton, G. N. 1984. Developments in the heat balance method for simulating room thermal response. In: Proceedings of the workshop on HVAC controls, modeling and simulation, Atlanta, GA, 2–3 February 1990. Georgia Institute of Technology. 18 p.

Özisik, M. N. 1968. Boundary value problems of heat conduction. Scranton, PA: International textbook company. 505 p.

APPENDIX 1

Convergence, consistency and stability of the solution

Convergence of the series solution

Solution (15) represents a functional series approaching infinite. The series may include both $a_n(x)$ and $b_n(x)$ functions. Each term of the series is defined in the interval $0 \le x \le L$. According to Carslaw (1963), a necessary and sufficient condition for the uniform convergence of the series in this interval is that, if any positive number ε has been chosen, as small as we please, there shall be a positive integer ν such that for all values of x in the interval, $|R_{p,n}(x)| < \varepsilon$, when $n \ge \nu$, for every positive integer p.

If a solution with $a_n(x) = 0$, $n = 0, 1, 2, ...$, and the first kind of boundary conditions equal to one at all time steps, is represented by a series

$$b_0(x) + b_1(x) + b_2(x) + b_3(x) + \cdots \qquad (1)$$

$R_{p,n}(x)$ is defined as

$$R_{p,n}(x) = b_{n+p}(x) - b_n(x). \qquad (2)$$

It is first assumed that $p = 1$ and $n + 1$ is an odd number. Then

$$\begin{aligned} R_{n+1,n}(x) = {} & (\mu_{n+1,1} - \mu_{n,1})\beta_1(x) + (\mu_{n+1,2} - \mu_{n,2})\beta_2(x) + \cdots \\ & + (\mu_{n+1,1} - \mu_{n,n})\beta_n(x) + \frac{1}{2}\mu_{n+1,n+1}\frac{(qx)^{2n+1}}{(2n+1)!}\cosh qx \end{aligned} \qquad (3)$$

where $\beta(x)$ refers to a bounded function combined of hyperbolic function and powers. $R_{n+1,n}(x)$ consists of two types of terms. The first type is expressions with $\beta(x)$-functions. They all include a difference of coefficients μ, approaching zero as n tends to infinite. So one can choose any number ε, for which there is a positive integer ν satisfying the required condition.

The last terms of $R_{n+1,n}(x)$ represents the second type of expression, for which one can make the same kind of conclusion. Because qx and $coshqx$ are bounded, the expression approaches zero when n tends to infinite. This is due to the increasing value of denominators.

The above case was only for $p = 1$. If $p > 1$ the expression of $R_{n+p,n}(x)$ includes several expressions of the second type. The condition is satisfied by all of them. The same is also true for terms with an even subscript $n + 1$. Similar deduction could be made for series of $a_n(x)$ functions.

Consistency and stability

The time derivative definition with the relationship between the displacement and the lumping constants: η, and h makes the solution (15) consistent with the corresponding analytical solution. Equation (15) converges to the analytical solution when time step approaches zero. The solution is also stable according to the Lax equivalence theorem (Strang 1986) stating that the combination of consistency and stability is equivalent to convergence. This theorem is valid for linear initial and boundary value problems and for approximation of functions.

References

Carslaw, H. S. 1963. Introduction to the theory of Fourier's series and integrals. New York, NY: Dover Publications. 368 p.

Strang, G. 1986. Introduction to applied mathematics. Wellesley, MA: Wellesley-Cambridge Press. 758 p.

APPENDIX 2

Computer programs for the explicit numerical method and the differential-difference method

```
      PROGRAM EXPLIC
C COMPUTER PROGRAM OF EXPLICIT NUMER. METHOD FOR RAMP
C RESPONSE, APPLIED TO ONE-LAYER, FINITE SLAB
      REAL T(0:30,2),L,M,DX,DIFFUS,CONDUC,TSTEP,U2
      READ(*,*) LOOPS,NPOINT,TSTEP,L,DIFFUS,CONDUC,U2
    3 NPOINT = NPOINT-1 ! NUMBER OF NODAL POINTS
      DX = L/NPOINT
      M = DX**2/(DIFFUS*TSTEP)
      TMP = 1.-2./M
      IF((TMP.LT.0.).AND.(NPOINT.GT.2)) GOTO 3
      IF((TMP.LT.0.).AND.(NPOINT.GT.1)) GOTO 100
      DO 5 I = 0,POINT ! HEAT FLOW RATE AS A FUNCTION OF TIME
    5 T(I,0) = 0.
      DO 10 I = 1,LOOPS
      T(0,1) = U2*I
      DO 20 N = 1,NPOINT-1
   20 T(N,1) = (T(N+1,0)+T(N-1,0))/M+TMP*T(N,0)
      DO 30 J = 0,POINT
   30 T(J,0) = T(J,1)
      P1 = - CONDUC*(T(NPOINT-1,1)-T(NPOINT,1))/DX
   10 WRITE(*,*) I,P1
  100 CONTINUE
      END
```

```
      PROGRAM DIFFER
C COMPUTER PROGRAM OF DIFF.-DIFF. METHOD FOR
C RAMP RESPONSE, APPLIED TO ONE-LAYER, FINITE SLAB.
      REAL BB(0:30),BD(0:30),P1(0:30),L,DIFFUS,CONDUC,U2,TSTEP
      READ(*,*) LOOPS,TSTEP,L,DIFFUS,CONDUC,U2,PAR
      KNUM = 0 ! INITIAL VALUES
      QL = L/SQRT(DIFFUS*TSTEP)
      BB(0) = SINH(QL)
      BD(0) = COSH(QL)
      DO 10 N = 1,20 ! COEFFICIENTS AND THEIR NUMBER
      BB(N) = ((2.*N-1.)*BB(N-1)-QL*BD(N-1))/(2.*N)
      BD(N) = -QL*BB(N-1)/(2.*N)
   10 IF((ABS(BB(N))).LT.PAR).AND.(KNUM.EQ.0)) KNUM = N
      DO 20 I = 0,KNUM ! HEAT FLOW RATE AS A FUNCTION OF TIME
   20 P1(I) = 0.
      TMP = BB(0)*L/(CONDUC*QL)
      DO 40 N = 1,LOOPS
      DO 30 I = KNUM,1,-1
      P1(0) = P1(0)-BB(I)/BB(0)*P1(I)
   30 P1(I) = P1(I-1)
      P1(1) = P1(0)-U2*N/TMP
      WRITE(*,*) N,P1(1)
   40 P1(0) = 0.
      END
```

APPENDIX 3

Pulse response factors of heavy and light multi-layer wall

The following numerical values illustrate coefficients of the differential-difference method. The coefficients are calculated using a time step of one hour and modified into the form of pulse transfer method of Equations (96) and (97). Also physical properties of the walls are shown.

Table 1. The pulse transfer coefficients of a heavy, multi-layer wall.

k	R_k	Y_k	Z_k	X_k
0	1.00000	0.00137	- 85.02454	- 85.02368
1	- 3.51869	0.00000	149.71950	149.71910
2	5.43269	0.00000	- 154.00100	- 154.00090
3	- 4.92849	0.00000	104.73000	104.73000
4	2.97607	0.00000	- 50.57932	- 50.57932
5	- 1.28541	0.00000	18.20159	18.20159
6	0.41667	0.00000	- 5.05638	- 5.05638
7	- 0.10498	0.00000	1.11458	1.11458
8	0.02112	0.00000	- 0.19933	- 0.19933
9	- 0.00347	0.00000	0.02946	0.02946
10	0.00047	0.00000	- 0.00365	- 0.00365
11	- 0.00005	0.00000	0.00039	0.00039
12	0.00001	0.00000	- 0.00003	- 0.00003

Table 2. The material properties of the heavy, multi-layer wall.

n	$L_n[m]$	$k_n[W/mK]$	$\rho_n[kg/m^3]$	$c_n[J/kgK]$	$1/h_s[m^2K/W]$
1	0.130	0.950	1923.0	920.0	0.00
2	0.175	0.045	30.0	840.0	0.00
3	0.020	0.000	0.0	0.0	0.17
4	0.130	0.950	1923.0	920.0	0.00

Table 3. Pulse transfer coefficients of a light, multi-layer wall.

k	R_k	Y_k	Z_k	X_k
0	1.00000	0.07030	- 5.01366	- 4.97891
1	- 0.98879	0.00000	3.03933	3.02442
2	0.39830	0.00000	- 0.90968	- 0.90707
3	- 0.08876	0.00000	0.16030	0.16006
4	0.01248	0.00000	- 0.01846	- 0.01845
5	- 0.00120	0.00000	0.00149	0.00149
6	0.00008	0.00000	- 0.00009	- 0.00009

Table 4. The material properties of the light, multi-layer wall.

n	$L_n[m]$	$k_n[W/mK]$	$\rho_n[kg/m^3]$	$c_n[J/kgK]$	$1/h_s[m^2K/W]$
1	0.020	0.140	460.0	1360.0	0.00
2	0.175	0.041	30.0	840.0	0.00
3	0.022	0.000	0.0	0.0	0.17
4	0.020	0.140	460.0	1360.0	0.00

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Conduction of heat through slabs and walls

A differential-difference approach for design, energy analysis and building automation applications

Abstract

Modelling of thermal behaviour of buildings needs effective tools. This is particularly true when conduction of heat through slabs and/or walls is computed. The paper proposes a novel approach for such applications. The method is based on differential equation of heat conduction which is further modified to a differential-difference equation with continuous space variable and discrete time variable. The approach differs from conventional differential-difference solutions. In this paper, one-dimensional problems are examined in semi-infinite, one- and multi-layer environment.

Characteristic of the method is that solutions are presented using past values of boundary functions. In addition, transfer functions which determine the response at each time instant are calculated recursively. Because the differential-difference solution is partly numerical, better accuracy is achieved by using analytical methods, such as the pulse transfer method. However, in a multi-layer environment the latter turns out to be more complicated, since several transcendental equations must be solved, contrary to the proposed method.

The differential-difference method is compared with numerical solutions choosing the explicit method as a representative of them. The results show that in most cases better accuracy is achieved with the differential-difference method when time steps of both methods are equal. In addition, the proposed method needs no nodal points inside the slab during computation. Thus, time steps need not be adjusted according to thin layers of the wall, which makes the method feasible in multi-layer environment. The differential-difference approach is inherently stable, which is not true for all numerical methods. The method is suggested to be applied in dynamic thermal models of buildings in which time step is less than one hour.

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