



Seppo Uosukainen

# Foundations of acoustic analogies



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## Abstract

This report presents the best-known acoustic analogies, and their equations are derived mathematically in detail to allow their applicability to be extended when necessary. In the acoustic analogies, the equations governing the flow-generated acoustic fields are rearranged in such a way that the field variable connections (wave operator part) are on the left-hand side and that which is supposed to form the source quantities for the acoustic field (source part) is on the right-hand side.

Lighthill's analogy was originally developed for unbounded flows. The analogy assumes that, outside the source region, there is no static flow and the fluid is ideal. The refraction effects are not included in the wave operator. Powell's analogy is an approximate version of Lighthill's analogy. The Ffowcs Williams–Hawkings analogy is such an extension of Lighthill's analogy that, being based on the same starting point, it takes into account the effects of moving boundaries by equivalent Huygens sources. Curle's analogy is obtained from the Ffowcs Williams–Hawkings analogy by assuming that the boundaries are not moving. In Phillips' analogy, the effects of a moving medium are partially taken into account, and the refraction effects are included in the wave operator. The fluid outside the source region is assumed to be ideal. Lilley's analogy is based on the same starting point as Phillips' analogy, but all the 'propagation effects' occurring in a transversely sheared mean flow are inside the wave operator part of the equation. In Howe's analogy, the vorticity vector (in the form of Coriolis acceleration) and the entropy gradients are put in the source part of the equation, forming the main part of the sources; the compressibility of the medium is assumed to be constant and the viscous losses are assumed to vanish. In Doak's analogy, the compressibility of the medium does not need to be constant, the vorticity and the entropy gradients do not need to disappear outside the source region, and the viscous and thermal losses can be taken into account, somehow, inside and outside the source region.

The four last-presented analogies assume that the medium is an ideal gas, so without modifications they cannot be applied to acoustic fields in liquids.

## Tiivistelmä

Tunnetuimmat akustiset analogiat esitetään ja niitä hallitsevat yhtälöt johdetaan matemaattisesti yksityiskohtaisesti, jotta niiden sovellettavuutta olisi mahdollista tarvittaessa laajentaa. Akustisissa analogioissa virtausherätteisiä akustisia kenttiä hallitsevat yhtälöt järjestellään siten, että vasemmalla puolella ovat kenttämuuttujayhteydet (aalto-operaattoriosat) ja oikealla puolella jotakin, jonka oletetaan muodostavan akustisen kentän lähdesuureet (lähdeosa).

Lighthillin analogia on alun perin kehitetty rajoittamattomille virtauksille. Analogiassa oletetaan, että lähdealueen ulkopuolella ei ole staattista virtausta ja että fluidi on ideaalinen. Aalto-operaattori ei sisällä aallon taittumisen vaikutuksia. Powellin analogia on approksimaatio Lighthillin analogiasta. Ffowcs Williamsin–Hawkinsin analogia on Lighthillin analogian laajennus siten että, perustuen samaan lähtökohtaan, se ottaa huomioon liikkuvien rajapintojen vaikutukset ekvivalenttisilla Huygensin lähteillä. Curlen analogia saadaan Ffowcs Williamsin–Hawkinsin analogiasta olettamalla rajapinnat liikkumattomiksi. Phillipsin analogiassa liikkuvan väliaineen vaikutukset otetaan osittain huomioon ja aallon taittumisen vaikutukset sisältyvät aalto-operaattoriin. Lähdealueen ulkopuolinen fluidi oletetaan ideaaliseksi. Lilleyin analogia perustuu samaan lähtökohtaan kuin Phillipsin analogia, mutta kaikki virtausten sekoittumisessa esiintyvät ”etenemisvaikutukset” ovat yhtälön aalto-operaattoripuolella. Howen analogiassa pyörteisyysektori (Coriolis-kiihtyvyyden muodossa) ja entropiagradienit on sijoitettu yhtälön lähdepuolelle niiden muodostaessa pääosan lähteistä, väliaineen puristuvuuden oletetaan olevan vakio sekä viskoottisia häviöitä ei oteta huomioon. Doakin analogiassa puristuvuuden ei tarvitse olla vakio, pyörteisyyden ja entropiagradienitien ei tarvitse hävitä lähdealueen ulkopuolella, ja viskoottiset ja lämpöhäviöt voidaan jollain lailla ottaa huomioon sekä lähdealueen sisä- että ulkopuolella.

Neljässä viimeiseksi esitetyssä analogiassa väliaine oletetaan ideaalikaasuksi, joten ilman modifikaatioita niitä ei voi soveltaa akustisiin kenttiin nesteissä.

## **Preface**

The work described in this publication has been carried out in VTT Smart Machines, Machinery and Environmental Acoustics Team. The results were obtained through the UNNO task (Underwater Noise) of the SEEE project (Ship's Energy Efficiency and Environment) in the EFFIMA programme (Energy and lifecycle efficient machines) of FIMECC SHOK with funding from Tekes. The programme runs from 2009 to 2013. The aim of this report is to present the best-known acoustic analogies and derive their equations mathematically in detail to clarify their applicability to calculating flow-generated acoustic fields and to allow their applicability to be extended when necessary.

Espoo, January 2011

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## List of symbols

|                                   |   |
|-----------------------------------|---|
| $B$                               | stagnation enthalpy, Eqs. (39) & (A.7)                                    |
| $\underline{\underline{C}}$       | Doppler factor, Eqs. (29) & (J.29)  |
| $\underline{\underline{E}}$       | rate-of-strain dyadic, Eq. (B.3)  |
| $E_{\text{int}}$                  | internal energy per unit mass   |
| $\vec{F}$                         | strength of force source distribution (dipole distribution + gravitation) |
| $\vec{F}_S$                       | surface force source distribution, Eq. (G.9)                              |
| $H$                               | enthalpy, Eqs. (40) & (A.6); Heaviside function (step function), Eq. (14) |
| $\mathbf{I}$                      | identical dyadic, Eq. (R.9)   |
| $K$                               | thermal conductivity of fluid   |
| $L$                               | wave operator   |
| $P$                               | pressure  |
| $P_0$                             | static pressure; constant reference pressure in the definition of $\Pi$   |
| $Q$                               | total volume velocity   |
| $R$                               | gas constant; distance between source and field points, Eq. (P.2)         |
| $S$                               | entropy; surface  |
| $T$                               | temperature; limit of time interval in App. P                             |
| $\underline{\underline{T}}$       | strength of momentum source distribution (quadrupole distribution)        |
| $\underline{\underline{T}}_L$     | Lighthill's stress dyadic, Eqs. (5) & (H.4)                               |
| $\underline{\underline{T}}_R$     | Reynolds' stress, Eq. (7)   |
| $\underline{\underline{T}}_S$     | surface quadrupole source distribution                                    |
| $\underline{\underline{T}}_{wvc}$ | volume quadrupole distribution substituting $q_{ws}$ , Eqs. (20) & (J.11) |
| $\vec{U}$                         | particle velocity   |
| $V$                               | volume  |

|                             |  |
|-----------------------------|--|
| $\bar{V}$                   | auxiliary variable in Doak's analogy, Eqs. (43) & (O.15)   |
| $V^*$                       | specific volume, Eq. (C.3)   |
| $\bar{a}$                   | acceleration   |
| $c$                         | local speed of sound in constant entropy, Eq. (A.4)  |
| $c_0$                       | linearized speed of sound  |
| $c_P$                       | specific heat in constant pressure   |
| $c_T$                       | isothermal speed of sound, Eq. (C.3)   |
| $c_V$                       | specific heat in constant volume   |
| $\bar{e}_n$                 | unit normal vector   |
| $\bar{e}_i$                 | unit normal vector in $w_i$ direction ( $i = 1, 2, 3$ )  |
| $\bar{e}_x$                 | unit normal vector in $x$ direction  |
| $\underline{\underline{e}}$ | perturbation part of rate-of-strain dyadic $\underline{\underline{E}}$   |
| $f$                         | field  |
| $\bar{f}$                   | strength of perturbation force source distribution (dipole distribution)   |
| $\bar{f}_S$                 | surface dipole source distribution, linearized part of $\bar{F}_S$ ; equivalent Huygens surface dipole source distribution, Eq. (G.13) |
| $\bar{f}_{ws}$              | equivalent Huygens surface dipole source distribution (loading source) for moving obstacles, Eqs. (16) & (J.3)                         |
| $\bar{f}_{wvc}$             | volume dipole distribution substituting $q_{ws}$ , Eqs. (19) & (J.10)  |
| $g$                         | source in Eq. (1); Green's function  |
| $\bar{g}$                   | acceleration of gravity  |
| $g_0$                       | Green's function for free space, Eq. (F.2)   |
| $h_i$                       | scale factors of coordinate system $w_i$ , $i = 1, 2, 3$ , Eq. (G.7)   |
| $p$                         | pressure perturbation (sound pressure)   |
| $p_f$                       | sound pressure due to dipole distribution, Eq. (F.8) for volume distribution, Eq. (F.12) for surface distribution                      |
| $p_{fS}$                    | sound pressure due to $\bar{f}_{ws}$ , Eq. (J.33)  |
| $p_{fvc}$                   | sound pressure due to $\underline{\underline{f}}_{wvc}$ , Eq. (J.31)   |
| $p_L$                       | sound pressure due to $T_L$ , Eq. (J.34)   |
| $p_q$                       | sound pressure due to monopole distribution, Eq. (F.7) for volume distribution, Eq. (F.11) for surface distribution                    |
| $p_T$                       | sound pressure due to quadrupole distribution, Eq. (F.9) for volume distribution, Eq. (F.13) for surface distribution                  |

|               |   |
|---------------|---|
| $p_{TVc}$     | sound pressure due to $\overline{T_{wvc}}$ , Eq. (J.32)   |
| $p_{vS}$      | sound pressure due to $q_{wS}$ , Eq. (J.30)   |
| $q$           | strength of mass source distribution (monopole distribution, volume velocity distribution)                        |
| $q_S$         | surface monopole source distribution, Eq. (G.8); equivalent Huygens surface monopole distribution, Eq. (G.12)     |
| $q_{wS}$      | equivalent Huygens surface monopole source distribution (thickness source) for moving obstacle, Eqs. (15) & (J.2) |
| $\vec{r}$     | field point vector  |
| $\vec{r}_0$   | source point vector   |
| $s$           | perturbation part of entropy $S$  |
| $s_e$         | entropy-related auxiliary variable, Eq. (6)   |
| $t$           | time  |
| $t_0$         | source time variable  |
| $t_e$         | retarded time, Eqs. (30) & (J.35)   |
| $\vec{u}$     | perturbation particle velocity of fluid, particle velocity associated with $p$                                    |
| $\vec{v}$     | velocity  |
| $w_i$         | coordinate ( $i = 1, 2, 3$ )  |
| $x$           | Cartesian coordinate  |
| $y$           | Cartesian coordinate  |
| $z$           | Cartesian coordinate  |
| $\Delta a$    | discontinuity in variable $a$   |
| $\Pi$         | scaled logarithmic pressure, Eqs. (35) & (C.12)   |
| $\Psi$        | effects of entropy fluctuations and fluid viscosity in Lilley's analogy, Eqs. (38) & (M.4)                        |
| $\beta$       | coefficient of thermal expansion  |
| $\delta$      | Dirac delta function  |
| $\varepsilon$ | energy per unit volume delivered by heat source distribution  |
| $\gamma$      | adiabatic constant  |
| $\mu$         | coefficient of viscosity  |
| $\mu_v$       | expansion coefficient of viscosity  |
| $\rho$        | density   |
| $\rho'_{fS}$  | density perturbation due to $\vec{f}_{wS}$ , Eq. (27)   |

|                                      |  |
|--------------------------------------|--|
| $\rho'_{fvc}$                        | density perturbation due to $\underline{\underline{f}}_{wvc}$ , Eq. (25) |
| $\rho'_L$                            | density perturbation due to $\underline{T}_L$ , Eq. (28)                 |
| $\rho'_{TVc}$                        | density perturbation due to $\underline{\underline{T}}_{wvc}$ , Eq. (26) |
| $\rho'_{vS}$                         | density perturbation due to $q_{wS}$ , Eq. (24)                          |
| $\underline{\underline{\sigma}}_\mu$ | viscous part of stress dyadic, Eq. (B.2)                                 |
| $\bar{\omega}$                       | vorticity distribution, Eqs. (11) & (B.8)                                |
| $\bar{\zeta}$                        | Lagrangian source point vector   |
| $\nabla_0$                           | operator $\nabla$ operating on source coordinates                        |
| o                                    | (subscript) static part  |
| T                                    | (subscript) transpose  |
| '                                    | (superscript) perturbation part  |
| —                                    | (overline) temporal mean value   |
| () <sub>f</sub>                      | meaning: quantity $f$ is constant in the operation inside brackets       |

# 1. Introduction

This report presents the best-known acoustic analogies, and their equations are derived mathematically. The acoustic analogies are used to describe the connection between the flow and the sound field due to the flow, i.e., the dependence of the flow-generated sound on its causes (sources). Here, the analogies are divided into three categories: density-based, phi-based and enthalpy-based analogies. The first two names of the categories used here are not in general use. The phi-based analogies are called convected wave equation analogies by Karjalainen [1]. The division used here is based on the principal acoustic field variable used in the analogies. Lighthill's analogy, Powell's analogy, the Ffowcs Williams–Hawkings analogy and Curle's analogy are assigned to density-based analogies, Phillips' analogy and Lilley's analogy to phi-based analogies, and Howe's analogy and Doak's analogy to enthalpy-based analogies. The selection of analogies to be studied is based principally on a VTT Report by Karjalainen [1]. The mathematical complexity of applying these analogies grows in the same order as they are presented here [1], except for Powell's and Curle's analogies, which are special cases of Lighthill's analogy and the Ffowcs Williams–Hawkings analogy.

In the acoustic analogies, the equations governing the acoustic fields are rearranged in such a way that the field variable connections (wave operator part) are on the left-hand side and that which is supposed to form the source quantities to the acoustic field (source part) is on the right-hand side, as

$$Lf = g, \tag{1}$$

where  $Lf$  is the wave operator part containing operator  $L$  and field  $f$  to be calculated, and  $g$  is the sources for field  $f$ . Outside the source region, the right-hand side of the equation is zero and the field there obeys the homogeneous wave equation. The right-hand side sources have to be known *a priori* or the field

## 1. Introduction

equation should be solved iteratively the source part becoming more accurate at every iteration loop.

Apart from being based on different field variables, the various analogies differ from each other also with respect to the terms in the equations that are defined to form the right-hand side source quantities and the terms that are defined as belonging to the left-hand side, describing the behaviour of the field variable. There is no unique truth for the justifications of these rearrangements; they can rather be said to be matters of opinion. The source terms are, somehow, mostly collected from the field variable terms, only some analogies use true acoustic source distributions as parts of sources.

In this text, the true acoustic sources (mass, heat, force or momentum sources) are defined as distributions producing acoustic energy. The internal losses (viscous and thermal) are defined as sinks reducing acoustic energy. Thus, the true source and loss concepts are separated. E.g., the heat sources are producing and thermal losses are dissipating acoustic energy. In most of the acoustic analogies presented here, the loss terms are placed on the source part of the equation.

The dependence of the generated sound power on the flow velocity depends on the source part formulation. This topic is beyond the scope of this report. Some aspects on this have been presented, e.g., in Ref. [1].

The equations governing the analogies are derived in detail from four principal non-linearized equations of acoustics for Newtonian fluids: the equation of continuity, the Navier-Stokes equation, the state equation and the energy equation. Although references are given for each derivation, the derivations in the references are not as detailed as in this publication: many intermediate grades are skipped in the references mentioned. Some derivations are deduced here in a slightly different way to that in the references. Contrary to a typical reference, assumptions that have to be made to obtain the final results are also introduced as late as possible in the derivations. This kind of detailed approach has been selected here to clarify what must be assumed and at what stage, to obtain the required equations. This facilitates to extend further the applicability of the analogies when necessary.

The principal equations and all the derivations of the required equations are presented in the appendices. The main text only introduces the final versions of the analogies, the basic starting points of the analogies and all the assumptions needed in their derivations.

In many cases, the concepts of ideal gas and ideal fluid are used. The concepts are not the same. With an ideal gas, there are no force interactions between the

gas molecules and, mathematically, the gas obeys the simple state equation of the ideal gas. With gases, the deviation from an ideal gas tends to decrease with higher temperature and lower density. An ideal fluid can be a gas or a liquid such that there are no viscous or thermal losses.

The dyadic notation is used instead of the tensor notation, although the tensor notation is more widely used in the references. This choice was made because, with the dyadic notation, the formulae are, in most cases, much simpler and more illustrative than with the tensor notation, at least for the author. A short presentation of the basics of the dyadic notation is presented in Appendix R.

In the following text, the Lagrange description and the Euler description of motion are both used. The Lagrange description describes the motion of individual particles and the Euler description describes the motion at fixed points in space. The time derivative in the Lagrange description is expressed with the total derivative  $d/dt$ , and in the Euler description with the partial derivative  $\partial/\partial t$ . Their connections with scalar function  $f$ , vector function  $\vec{f}$  and dyadic function  $\underline{\underline{f}}$  are [2]

$$\begin{aligned}\frac{d f}{d t} &= \frac{\partial f}{\partial t} + \vec{U} \cdot \nabla f \\ \frac{d \vec{f}}{d t} &= \frac{\partial \vec{f}}{\partial t} + \vec{U} \cdot \nabla \vec{f} \\ \frac{d \underline{\underline{f}}}{d t} &= \frac{\partial \underline{\underline{f}}}{\partial t} + \vec{U} \cdot \nabla \underline{\underline{f}},\end{aligned}\tag{2}$$

where  $\vec{U}$  is the particle velocity.

## 2. Density-based analogies

The density-based analogies use  $\rho'$ , the perturbation component of the density  $\rho$ , or the pressure perturbation (sound pressure)  $p$ , as the basic field quantity, see Appendix E for the perturbation quantities in the linearization of the acoustic field equations. Either of these can be used if the perturbation entropy variations are assumed to be small, in which case, according to Eq. (E.8), they have a connection

$$\rho' = \frac{p}{c_0^2}, \quad (3)$$

where  $c_0$  is the linearized (ambient) speed of sound.

Lighthill's analogy, Powell's analogy, the Ffowcs Williams–Hawkins analogy, and Curle's analogy are presented in this category.

### 2.1 Lighthill's analogy

Lighthill published the first articles on flow-generated sound in 1952 and 1954 [3, 4]. These articles are seen as the birth of aeroacoustics [1]. Lighthill's analogy was originally developed for unbounded flows due to, e.g., old jet engines, without any heat sources.

Lighthill's equation is derived in Appendix H, Eq. (H.3), and it is [3, 4, 5]

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = \nabla \nabla : \overline{T_L}, \quad (4)$$



where  $c$  is the local speed of sound in constant entropy, defined in Eq. (A.4),  $t$  is time, and the Lighthill's turbulence stress dyadic (or traditionally tensor) is, see Eq. (H.4),

$$\overline{\overline{T}}_L = \rho \overline{\overline{U}} \overline{\overline{U}} + s_e \overline{\overline{I}} - \overline{\overline{\sigma}}'_\mu, \quad (5)$$

where  $\overline{\overline{U}}$  is the particle velocity,  $\overline{\overline{I}}$  is the identic dyadic,  $\overline{\overline{\sigma}}'_\mu$  is the perturbation component of the viscous part of the stress dyadic  $\sigma_\mu$ , the latter being defined in Eq. (B.2), and where  $s_e$  is an entropy-related auxiliary variable

$$s_e = p - c^2 \rho' = \frac{\rho c^2 \beta T}{c_p} s, \quad (6)$$

where  $s$  is the perturbation part of the entropy  $S$ , see Eq. (E.8),  $p$  is the pressure perturbation (sound pressure),  $T$  is the temperature,  $c_p$  is the specific heat at constant pressure, and  $\beta$  is the coefficient of thermal expansion. The first term in Lighthill's stress dyadic

$$\overline{\overline{T}}_R = \rho \overline{\overline{U}} \overline{\overline{U}} \quad (7)$$

is called Reynolds' stress. The second term is related to the entropy changes in acoustic fields and the third term to the viscotic shear stresses caused by gradients of the acoustic particle velocity.

The convective velocity terms are all put into the source part of the equation in Lighthill's analogy, so it is assumed in the analogy that the static flow velocity is zero outside the source region. The entropy fluctuations and the viscous stresses are also put into the source part, so it is assumed that there are no losses due to the viscosity or the thermal conductivity of the fluid outside the source region, which means that the fluid outside the source region is an ideal fluid. The entropy fluctuation term also contains the dependence of the sound speed on the spatial coordinates, so the sound speed should be constant with respect to the spatial coordinates outside the source region. The refraction effects are therefore not included in the wave operator.

Assumptions that have been made to obtain Eq. (4):

## 2. Density-based analogies

- The field quantities have to be such that they can be divided into static and time-dependent perturbation components. The perturbation components have to be much smaller than the static ones so that the static components do not depend on the perturbation components.
- The sound speed and the static parts of the pressure, density, entropy and particle velocity must not change very much as functions of the spatial coordinates: their gradients should be, at most, of the perturbation order. The sound speed is not a function of time.
- There are no mass, heat, force or momentum source distributions.

As one starting point for Lighthill's analogy is that the static flow velocity is zero and the sound speed is constant outside the source region, the assumption that the sound speed and the static part of the particle velocity must not change very much as a function of the spatial coordinates concerns the source region.

If Lighthill's stress dyadic in Eq. (4) is compared with the last term on the right-hand side of Eq. (H.1), Lighthill's stress dyadic can be seen to be formally similar to a quadrupole source distribution. Its most important part is Reynolds' stress, especially in isentropic flows, which implicates that the spatial particle velocity fluctuations are the main source of flow-generated sound. The other terms,  $\sigma'_{\mu}$  and  $s_e \mathbf{I}$ , in Lighthill's stress dyadic concern the viscous and thermal losses inside the source region.

Uosukainen [6] has suggested that the terms  $\overline{\sigma'_{\mu}}$  and  $\overline{s_e \mathbf{I}}$  in Lighthill's stress dyadic should be moved to the left-hand side of the equation to allow the viscous and thermal losses and the dependence of the sound speed on the spatial coordinates outside the source region to be taken into account. In that case Lighthill's stress dyadic is the same as Reynolds' stress in Eq. (7). He further suggests that in Reynolds' stress, all purely static terms and the terms containing irrotational perturbation velocity and density perturbation should be excluded from the source part and moved to the left-hand side of the equation.

If the interaction between the sound field and the mean flow (including the convection and the refraction) is to be taken into account, the source term must be adjusted, as these effects are included there. This cannot be done until after the equation has been solved [5].

Assuming that the space to be handled is free and the perturbation entropy variations are small, the density perturbation can be obtained from Eq. (H.6) when transformed into a density quantity as [5]

$$\rho'(\vec{r}, t) = \frac{1}{c_0^2} \nabla \nabla : \int_V \frac{\overline{\overline{T_L}}(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dV, \quad (8)$$

where  $c_0$  is the linearized speed of sound,  $\vec{r}$  is a field point vector,  $\vec{r}_0$  is a source point vector, the volume integration is performed with respect to the  $\vec{r}_0$  coordinates, and  $V$  is the volume at which Lighthill's stress dyadic is non-zero (practically the volume that contains all regions in which Lighthill's stress dyadic is significant).

## 2.2 Powell's analogy

Powell's analogy is an approximate version of Lighthill's analogy and is based on the same starting point and basic assumptions as the latter.

Powell's equation is derived in Appendix I, Eq. (I.2), and it is [7]

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = -\nabla \cdot \vec{f}_P, \quad (9)$$

where the source function, replacing Lighthill's stress dyadic, is, see Eq. (I.3),

$$\vec{f}_P = -\left[\rho \vec{\omega} \times \vec{U} + \frac{1}{2} \nabla (\rho \vec{U} \cdot \vec{U})\right], \quad (10)$$

where  $\vec{\omega}$  is the vorticity distribution according to Eq. (B.8)

$$\vec{\omega} = \nabla \times \vec{U} \quad (11)$$

and term  $-\vec{\omega} \times \vec{U}$  is the Coriolis acceleration [8, 9]. In this case, the flow-generated sound can be seen to be due mainly to the vorticity, more specifically to the changes of the Coriolis acceleration, in the source region [7, 10, 11]. Powell has argued that the vorticity usually has the predominant role in the aerodynamic generation of sound [7]. The flow-generated noise in the ducts, in particular, is seen as being mostly due to the vorticity [1]. In the potential flow approaches, the vorticity is taken as the aerodynamic source of sound [1].

If the source function in Eq. (9) is compared with the second to last term on the right-hand side of Eq. (H.1), it can be interpreted as a dipole distribution.

## 2. Density-based analogies

Assumptions other than those in Lighthill's analogy that have been made to obtain Eq. (9):

- There are no viscous or thermal losses (the fluid is ideal) also inside the source region.
- The fluid is incompressible inside the source region.

Assuming the space to be handled is free and the perturbation entropy variations are small, the density perturbation can be obtained from Eq. (I.5) when transformed into a density quantity as

$$\rho'(\vec{r}, t) = -\frac{1}{c_0^2} \nabla \cdot \int_V \frac{\vec{f}_p(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dV. \quad (12)$$

### 2.3 The Ffowcs Williams–Hawkings analogy

The Ffowcs Williams–Hawkings analogy is such an extension of Lighthill's analogy that it takes into account the effects of moving boundaries by equivalent Huygens sources consisting of surface monopole source distribution  $q_{ws}$  (thickness source) and surface dipole source distribution  $\vec{f}_{ws}$  (loading source). Thus, it is based on the same starting point and assumptions and equations as Lighthill's analogy, the expressions for Huygens sources included. The main aim is to handle solid surface interactions that are directly involved in the generation of flow sound, e.g., by helicopter rotors, aeroplane (or marine) propellers, and aircraft engine fans, compressors and turbines [5]. The Ffowcs Williams–Hawkings analogy therefore has considerably wider exploitation potential than the previous analogies.

Consider a body with volume  $V_c$  and outer surface  $S$  moving in space and let the rest space volume, with  $V_c$  excluded, be denoted by  $V$ ; see Figure 1.

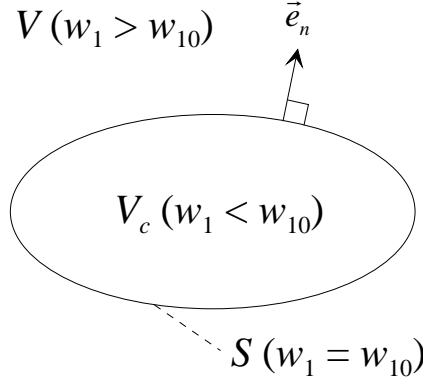


Figure 1. Moving volume  $V_c$  with surface  $S$ , outer volume  $V$ .

In this situation, the Ffowcs Williams–Hawkins equation has been derived in Appendix J, Eq. (J.1), and it is for the density perturbation  $\rho'$  in  $V$  [12, 13, 14]

$$\begin{aligned} \frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = & \frac{\partial [q_{ws} \delta(w_1 - w_{10})]}{\partial t} - \nabla \cdot [\vec{f}_{ws} \delta(w_1 - w_{10})] \\ & + \nabla \nabla : \left[ \overline{\overline{T_L H}}(w_1 - w_{10}) \right], \end{aligned} \quad (13)$$

where the equivalent surface distributions are on the surface  $S$  with  $w_1 = w_{10}$  in a coordinate system  $(w_1, w_2, w_3)$  such that  $V_c$  is the region in which  $w_1 < w_{10}$ ,  $\delta$  is the Dirac delta function, and  $H$  is the Heaviside function (step function):

$$H(x - x_0) = \begin{cases} 0, & x < x_0 \\ 1, & x > x_0. \end{cases} \quad (14)$$

Lighthill's stress dyadic  $\overline{\overline{T_L}}$  is presented in Eq. (5), and the equivalent source distributions are, see Eqs. (J.2) and (J.3),

$$q_{ws} = [\rho_0 \vec{v} + \rho(\vec{u} - \vec{v})] \cdot \vec{e}_n \quad (15)$$

$$\vec{f}_{ws} = p \vec{e}_n + \rho \vec{u}(\vec{u} - \vec{v}) \cdot \vec{e}_n - \vec{e}_n \cdot \overline{\overline{\sigma'}}_\mu, \quad (16)$$

## 2. Density-based analogies

where  $\vec{e}_n$  is a unit normal vector at  $S$  pointing outwards from  $V_c$ ,  $\vec{v}$  is the velocity of surface  $S$ ,  $\vec{u}$  is the perturbation particle velocity of the fluid, and  $\rho_0$  is the static density.

The basic assumptions that have been made to obtain Eq. (13) are the same as those made to obtain Lighthill's equation (4).

In most practical cases, the body surface  $S$  is impermeable and the normal components of the velocity of the surface and the fluid then coincide at the surface yielding to

$$q_{WS} = \rho_0 \vec{v} \cdot \vec{e}_n \quad (17)$$

$$\vec{f}_{WS} = p \vec{e}_n - \vec{e}_n \cdot \overline{\overline{\sigma'}}_{\mu}; \quad (18)$$

see Eqs. (J.6) and (J.7). The normal component of the surface velocity forms the equivalent surface monopole distribution, and the sound pressure and, in the case of the medium having viscous losses, the viscous part of the stress dyadic form the equivalent surface dipole source distribution, so these field quantities have to be known to take the effects of the surface into account.

In the case of the deformation of  $V_c$  being incompressible, leading to the situation in which the total volume  $V_c$  remains constant, it is not reasonable to use the surface monopole distribution  $q_{WS}$  and it can be replaced by volume dipole distribution  $\vec{f}_{wVc}$  and volume quadrupole distribution  $\overline{\overline{T}}_{wVc}$  inside  $V_c$ , where

$$\vec{f}_{wVc} = \rho_0 \vec{a} \quad (19)$$

$$\overline{\overline{T}}_{wVc} = \rho_0 \vec{v} \vec{v}, \quad (20)$$

where  $\vec{v}$  and  $\vec{a}$  are the velocity and acceleration in the Lagrangian coordinate system (velocity and acceleration of the individual particles) inside  $V_c$ ; see Eqs. (J.10) and (J.11) [12, 5]. In this case, Eq. (13) for the density perturbation fields in  $V$  can now be written as, see Eq. (J.12),

$$\begin{aligned}
 \frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' &= -\nabla \cdot \left\{ \vec{f}_{wvc} [1 - H(w_1 - w_{10})] \right\} \\
 &+ \nabla \nabla : \left\{ \overline{\overline{T}}_{wvc} [1 - H(w_1 - w_{10})] \right\} \\
 -\nabla \cdot \left[ \vec{f}_{ws} \delta(w_1 - w_{10}) \right] &+ \nabla \nabla : \left[ \overline{\overline{T}}_L H(w_1 - w_{10}) \right].
 \end{aligned} \tag{21}$$

In this approach, the particle velocity and acceleration distributions inside volume  $V_c$  have to be known, instead of the normal component of the surface velocity.

If volume  $V_c$  is a rigid body retaining its shape, and its movement consists of translation and rotation, the radiated linearized density perturbation can be presented using differential equation (13) as

$$\rho' = \rho'_{vs} + \rho'_{fs} + \rho'_L, \tag{22}$$

see Eq. (J.18), and using differential equation (21) as

$$\rho' = \rho'_{fvc} + \rho'_{Tvc} + \rho'_{fs} + \rho'_L, \tag{23}$$

see Eq. (J.13), where the partial density perturbation components due to different sources are as per Eqs. (J.30)–(J.34) [5, 12, 13]

$$\rho'_{vs}(\vec{r}, t) = \frac{1}{c_0^2} \frac{\partial}{\partial t} \int_S \frac{\rho_0 q_{ws}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dS(\vec{\zeta}) \tag{24}$$

$$\rho'_{fvc}(\vec{r}, t) = -\frac{1}{c_0^2} \nabla \cdot \int_{V_c} \frac{\vec{f}_{wvc}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dV(\vec{\zeta}) \tag{25}$$

$$\rho'_{Tvc}(\vec{r}, t) = \frac{1}{c_0^2} \nabla \nabla : \int_{V_c} \frac{\overline{\overline{T}}_{wvc}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dV(\vec{\zeta}) \tag{26}$$

$$\rho'_{fs}(\vec{r}, t) = -\frac{1}{c_0^2} \nabla \cdot \int_S \frac{\vec{f}_{ws}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dS(\vec{\zeta}) \tag{27}$$

## 2. Density-based analogies

$$\rho'_L(\vec{r}, t) = \frac{1}{c_0^2} \nabla \nabla : \int_V \frac{\overline{T}_L(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dV(\vec{\zeta}), \quad (28)$$

where  $C$  is the Doppler factor, see Eq. (J.29),

$$C(\vec{r}, \vec{\zeta}, t_e) = \left| 1 - \frac{\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)}{|\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)|} \cdot \frac{\vec{v}(\vec{\zeta}, t_e)}{c_0} \right| \quad (29)$$

and where the retarded time  $t_e = t_e(\vec{r}, t, \vec{\zeta})$  is the solution to the equation

$$h(t_e, t, \vec{r}, \vec{\zeta}) = t_e - t + \frac{|\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)|}{c_0} = 0; \quad (30)$$

see Eq. (J.35).

The integrals above are presented in a moving Cartesian coordinate system with vectors  $\vec{\zeta}(\vec{r}_0, t_0)$  such that the body will not move in this new coordinate system, i.e., in Lagrangian coordinates. The terms  $\vec{v}$  and  $\vec{a}$  in this coordinate systems signify the convectonal velocity and acceleration, i.e., the velocity and acceleration of individual points when the coordinate movement is temporally, fictitiously stopped.

If there is more than one solution to Eq. (30), a sum of all such solutions has to be used. This is the case at supersonic speeds. At subsonic speeds,  $S^*$ ,  $V_c^*$  and  $V^*$  are the physical surface and volumes  $S$ ,  $V_c$  and  $V$ ; see Figure 1. At supersonic speeds, they are functions of  $\vec{r}$  and  $t$  because the whole physical regions do not contribute to the integrals (Eq. (30) has no solutions for some values of  $\vec{\zeta}$ ).

For speeds at which the quantity  $C$  approaches one ( $v/c_0 \approx 0$ ), the solution fails due to singularities in the integrals. Other types of coordinate transformations are available to overcome this problem [12, 13].

In the derivation of the integrals above, it has been assumed that all moving or stationary surfaces are taken into account and that the perturbation entropy fluctuations are small.

If the primary sound source is the moving boundary, the effects of Lighthill's stress dyadic  $T_L$  on the sound radiation is typically lower than the effects of the moving boundaries if the velocity  $\vec{v}$  is low enough (below  $0.7 \cdot c_0$ ) [1]. This typically happens with, e.g., with helicopter rotors [13] and marine propellers [14].



The Ffowcs Williams–Hawkings analogy can be modified in such a way that stationary boundaries in a moving media can be handled. The flow-generated sound of surfaces guiding the flow can then be treated with the analogy.

## 2.4 Curle's analogy

Curle's analogy is a special version of the Ffowcs Williams–Hawkings analogy, and it is based on the same starting point, basic assumptions and equations as the latter. If the surface  $S$  is rigid and does not move, the equivalent monopole surface source distribution  $q_{WS}$  and its substitute volume dipole and quadrupole source distributions  $\vec{f}_{WVc}$  and  $\overline{T}_{WVc}$  disappear as well as the corresponding density perturbation components  $\rho'_{vS}$ ,  $\rho'_{fVc}$  and  $\rho'_{TVc}$ . This leads to Curle's equation in which the equivalent dipole surface source distribution  $\vec{f}_{WS}$  takes care of the sound scattering caused by the stationary surface.

Curle's equation is derived in Appendix K, Eq. (K.1), and it is for  $\rho'$  in  $V$  [15]

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = -\nabla \cdot \left[ \vec{f}_{WS} \delta(w_1 - w_{10}) \right] + \nabla \nabla : \left[ \overline{T}_L H(w_1 - w_{10}) \right], \quad (31)$$

where there are only two source functions, Lighthill's stress dyadic  $\overline{T}_L$  according to Eq. (5) and the equivalent dipole distribution  $\vec{f}_{WS}$  according to Eq. (18).

The radiated linearized density perturbation can be presented as

$$\rho' = \rho'_{fS} + \rho'_L, \quad (32)$$

see Eq. (K.4), where the partial density perturbation components due to the two sources according to Eqs. (K.7) and (K.8) are [15]

$$\rho'_{fS}(\vec{r}, t) = -\frac{1}{c_0^2} \nabla \cdot \int_S \frac{\vec{f}_{WS}(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c_0)}{4\pi|\vec{r} - \vec{r}_0|} dS \quad (33)$$

$$\rho'_L(\vec{r}, t) = \frac{1}{c_0^2} \nabla \nabla : \int_V \frac{\overline{T}_L(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c_0)}{4\pi|\vec{r} - \vec{r}_0|} dV. \quad (34)$$

### 3. Phi-based analogies

The phi-based analogies use the pressure-related field quantity  $\Pi$ , scaled logarithmic pressure, defined by Eq. (C.12), as the basic field quantity

$$\Pi = \frac{1}{\gamma} \ln \left( \frac{P}{P_0} \right), \quad (35)$$

where  $P$  is the pressure,  $P_0$  is some convenient constant reference pressure and  $\gamma$  is the adiabatic constant. Phillips' analogy and Lilley's analogy are presented in this category.

#### 3.1 Phillips' analogy

The Phillips' equation is derived in Appendix L, Eq. (L.2), and it is [16, 5]

$$\frac{d^2 \Pi}{dt^2} - \nabla \cdot (c^2 \nabla \Pi) = (\nabla \vec{U})_T : (\nabla \vec{U}) + \frac{d}{dt} \left( \frac{1}{c_p} \frac{dS}{dT} \right) - \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \vec{\sigma}_\mu \right), \quad (36)$$

where  $S$  is the entropy and subscript T denotes the transpose of a dyadic, see Eq. (R.11).

In Phillips' analogy, a convective term in the basic equations has been moved to the left-hand side of the equation. This leads to the total time derivative (of the second order) on the wave operator side, according to the Lagrange description of motion, taking into account, at least partially, the effects of the static flow [5]. Thus, the equation is valid for a moving medium, with some accuracy. The dependence of the sound speed on spatial coordinates has also been moved to the left-hand side of the equation, so the sound speed may be a function of coordinates outside the source region, and the refraction effects are included in the

wave operator. The entropy term and the viscous stresses are put into the source part, so it is assumed that there are no losses due to viscosity or the thermal conductivity of the fluid outside the source region, which means that the fluid outside the source region is assumed to be ideal. The entropy term also contains the effects of heat sources, according to Eq. (D.1), so these sources can be taken into account.

Assumptions that have been made to obtain Eq. (36):

- The medium considered is an ideal gas.
- There are no mass, force or momentum source distributions.

### 3.2 Lilley's analogy

Lilley published his analogy, which has been developed for the flow-generated noise of jet engines with a high by-pass ratio, in 1974 [17].

Lilley noticed that in Phillips' equation (36), the first source term on the right-hand side contains first order terms that should be included in the convective terms on the left-hand side of the equation [17]. He therefore derived the equation with respect to time in the Lagrangian way to obtain the terms outside the source terms and put them on the left-hand side of the equation. Otherwise, Lilley's analogy is based on the same starting point, assumptions and equations as Phillips' analogy, Lilley's equation just takes the effects of the static flow into account in a better way than the Phillips' equation.

Lilley's equation is derived in Appendix M, Eqs. (M.5) and (M.4), and it is [17, 5]

$$\begin{aligned} \frac{d}{dt} \left[ \frac{d^2 \Pi}{dt^2} - \nabla \cdot (c^2 \nabla \Pi) \right] + 2(\nabla \vec{U})_T : \nabla (c^2 \nabla \Pi) \\ = -2(\nabla \vec{U})_T : [(\nabla \vec{U}) \cdot (\nabla \vec{U})] + \Psi, \end{aligned} \quad (37)$$

where

$$\Psi = 2(\nabla \vec{U})_T : \nabla \left( \frac{1}{\rho} \nabla \cdot \vec{\sigma}_\mu \right) - \frac{d}{dt} \left[ \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \vec{\sigma}_\mu \right) \right] + \frac{d^2}{dt^2} \left( \frac{1}{c_p} \frac{dS}{dT} \right) \quad (38)$$

represents the effects of the entropy fluctuations and the gas viscosity.

### 3. Phi-based analogies

In Lilley's analogy, all the 'propagation effects' that occur in a transversely sheared mean flow are inside the wave operator part of the equation. In the case of parallel or nearly parallel mean flows (such as those that occur in by-pass jet engines and axial-flow fans), at least, no inconsistency is obtained when interpreting the right-hand side as sources [5].

The inclusion of the convection and refraction effects in the wave operator greatly increases the complexity of the solutions. In practice, this turns out to be a serious drawback, and only limited solutions to Lilley's and Phillips' analogies have been found [5]. The phi-based analogies have been used as a starting point in the aeroacoustics of jet engines. These analogies have actually not been applied to duct acoustics [1].

## 4. Enthalpy-based analogies

The enthalpy-based analogies use the stagnation enthalpy  $B$  as the basic field quantity, defined by Eq. (A.7) as

$$B = H + \frac{1}{2} \vec{U} \cdot \vec{U}, \quad (39)$$

where enthalpy  $H$  is defined by its difference in Eq. (A.6) as

$$dH = \frac{dP}{\rho} + T dS, \quad (40)$$

where  $S$  is the entropy.

### 4.1 Howe's analogy

Howe's equation is derived in Appendix N, Eq. (N.13), and it is [10]

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\ - \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{1}{c_p} \frac{dS}{dt} \right), \end{aligned} \quad (41)$$

where  $\vec{\omega}$  is the vorticity distribution and the term  $-\vec{\omega} \times \vec{U}$  is the Coriolis acceleration, see Eq. (11).

In Howe's analogy, the vorticity vector (in the form of Coriolis acceleration) and the entropy gradients are put in the source part of the equation, so the anal-

#### 4. Enthalpy-based analogies

ogy assumes that the vorticity and the entropy gradients disappear outside the source region and that the sources are mainly due to these quantities. The entropy time derivatives are also put into the source part, so it is assumed that there are no temporal entropy fluctuations except those connected to the entropy part of the stagnation enthalpy outside the source region. The thermal losses in the source region and, somehow, outside it can therefore be taken into account. The time-derivated entropy terms in the source part further ensure that the heat sources can be taken into account, according to Eq. (D.1).

Assumptions that have been made to obtain Eq. (41):

- The compressibility of the medium is constant with respect to time and spatial coordinates.
- There are no mass, force or momentum source distributions.
- There are no viscous losses.
- The medium considered is an ideal gas.

Howe's analogy is heavy in a computational sense and it is not widely used for estimating flow noise [1].

#### 4.2 Doak's analogy

Doak's equation is derived in Appendix O, Eqs. (O.14) and (O.15), and it is [8, 9]

$$\begin{aligned}
 & \nabla^2 B' - \left\{ \frac{1}{c^2} \left[ \frac{\partial^2}{\partial t^2} + \bar{U}\bar{U} : \nabla\nabla + \left( 2\bar{U} \frac{\partial}{\partial t} + \bar{\omega} \times \bar{U} + \bar{V} - 2\nabla H \right) \cdot \nabla \right] B' \right\}' \\
 & = \left[ \left\{ -(\nabla \cdot) + \frac{1}{c^2} \left[ \bar{U}\bar{U} : \nabla + (\bar{\omega} \times \bar{U} + \bar{V} - 2\nabla H) \cdot \right] \right\} \left[ (\bar{\omega} \times \bar{U})' - \bar{V}' - \overline{\left( \frac{\partial \bar{U}'}{\partial t} \right)} \right] \right]' \quad (42) \\
 & \quad + \left[ \frac{\partial}{\partial t} \left( \frac{1}{c^2} \right)' \frac{dH}{dt} \right]' - \left[ \frac{\partial}{\partial t} \left( \frac{1}{R} \frac{dS}{dt} \right)' \right]' - \left[ \frac{1}{c^2} \bar{U} \cdot \frac{\partial \bar{V}'}{\partial t} \right]',
 \end{aligned}$$

where overline means temporal mean value, superscript ' means the fluctuating (perturbation) part,  $R$  is the gas constant and

$$\vec{V} = T\nabla S + \frac{1}{\rho} \left( \nabla \cdot \overline{\sigma}_\mu + \vec{F} \right), \quad (43)$$

where  $\vec{F}$  denotes force source distribution.

In Doak's analogy, the compressibility of the medium does not need to be constant, and the vorticity and the entropy gradients do not need to disappear outside the source region. The entropy time derivative is put in the source part of the equation of Doak's analogy, so it is assumed that there are no temporal entropy fluctuations outside the source region except those connected to the entropy part of the stagnation enthalpy outside the source region. The viscous and thermal losses can thus be taken into account, somehow, inside and outside the source region, and the heat sources can be included.

Assumptions that have been made to obtain Eq. (42):

- The field quantities have to be such that they can be divided into static and time-dependent perturbation components. The perturbation components have to be much smaller than the static ones so that the static components do not depend on the perturbation components.
- There are no mass or momentum source distributions.
- The medium considered is an ideal gas.

Doak's analogy has not been used in real applications [1].

Both Howe's analogy and Doak's analogy give confirmation to Powell's identification [7] that the vorticity usually has the predominant role in the aerodynamic generation of sound. The vorticity appears in the form of Coriolis acceleration in the source terms of these analogies, as well as in Powell's analogy.

## 5. Summary

This report presents the best-known acoustic analogies, and their equations have been derived mathematically in detail to allow their applicability to be extended when necessary. The analogies have been divided into three categories: density-based, phi-based and enthalpy-based analogies. The division is based on the principal acoustic field variable used in the analogies. Lighthill's analogy, Powell's analogy, the Ffowcs Williams–Hawkings analogy and Curle's analogy are assigned to density-based analogies, Phillips' analogy and Lilley's analogy to phi-based analogies, and Howe's analogy and Doak's analogy to enthalpy-based analogies. In the acoustic analogies, the equations governing the flow-generated acoustic fields are rearranged in such a way that the field variable connections (wave operator part) are on the left-hand side and that which is supposed to form the source quantities for the acoustic field (source part) is on the right-hand side. The mathematical complexity of applying these analogies grows in the same order as they are presented, except for Powell's and Curle's analogies, which are special cases of Lighthill's analogy and the Ffowcs Williams–Hawkings analogy.

Lighthill's analogy was originally developed for unbounded flows due to, e.g., old jet engines. In the analogy, it is assumed that, outside the source region, there is no static flow and the fluid is ideal (no viscous or thermal losses). The refraction effects are not included in the wave operator. Lighthill's stress dyadic, which forms the source part of the analogy, can be seen to be formally similar to a quadrupole source distribution. Its most important part is Reynolds' stress, which implicates that the spatial particle velocity fluctuations are the main source of flow-generated sound.

Powell's analogy is an approximate version of Lighthill's analogy that assumes further that the fluid is ideal also inside the source region and, as well in



the source region, the fluid is incompressible. This leads to a dipole-type source function, mainly due to the vorticity in the form of Coriolis acceleration.

The Ffowcs Williams–Hawkings analogy is such an extension of Lighthill’s analogy that, being based on the same starting point, it takes into account the effects of moving boundaries by equivalent Huygens sources. The main aim is to handle solid surface interactions that are directly involved in the generation of flow-generated sound, e.g., by helicopter rotors, aeroplane or marine propellers, and aircraft engine fans, compressors and turbines. The normal component of the surface velocity forms the equivalent surface monopole source distribution and the sound pressure at the boundary forms the equivalent surface dipole source distribution. In the case the deformation of the body inside the surface is incompressible, the information concerning the normal component of the surface velocity can be replaced by the velocity and acceleration information of the body, forming equivalent quadrupole and dipole volume source distributions.

Curle’s analogy is obtained from the Ffowcs Williams–Hawkings analogy by assuming that the boundaries are not moving. In this case, only the equivalent surface dipole source distribution is present to take into account the sound scattering due to the surface.

In Phillips’ analogy, a convective term in the basic equations has been moved to the left-hand side of the equation to take into account the effects of a moving medium. The refraction effects are included in the wave operator. The fluid outside the source region is assumed to be ideal. The heat sources can be taken into account.

Lilley’s analogy is based on the same starting point as Phillips’ analogy, the difference being that the first order terms in the source part that should be included in the convective terms have been moved to the left-hand side of the equation. In Lilley’s analogy, all the ‘propagation effects’ that occur in a transversely sheared mean flow are inside the wave operator part of the equation. In the case of parallel or nearly parallel mean flows (such as those that occur in bypass jet engines and axial-flow fans), at least, no inconsistency is obtained when interpreting the right-hand side as sources.

The inclusion of the convection and refraction effects in the wave operator greatly increases the complexity of the solutions, and only limited solutions of Lilley’s and Phillips’ analogies have been found. The  $\phi$ -based analogies have been used as a starting point in the aeroacoustics of modern jet engines. These analogies have actually not been applied to duct acoustics.

## 5. Summary

In Howe's analogy, the vorticity vector (in the form of Coriolis acceleration) and the entropy gradients are put in the source part of the equation, forming the main part of the sources. The heat sources can be taken into account. The compressibility of the medium is assumed to be constant and the viscous losses are assumed to vanish. Howe's analogy is heavy in the computational sense and it is not widely used for estimating flow noise.

In Doak's analogy, the compressibility of the medium does not need to be constant, and the vorticity and the entropy gradients do not need to disappear outside the source region. The viscous and thermal losses can be taken into account, somehow, inside and outside the source region, and the heat sources can be included. Doak's analogy has not been used in real applications.

All the  $\phi$ -based and enthalpy-based analogies presented assume the medium to be an ideal gas, so without modifications they cannot be applied to acoustic fields in liquids. Only the density-based analogies are suitable to be applied in liquids without modifications.

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# Appendix A: Equation of continuity

## Basic version

The non-linear equation of continuity is, see, e.g., [5]

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{U} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = \rho q, \quad (\text{A.1})$$

where  $\rho$  is the density of the fluid,  $\vec{U}$  is the particle velocity,  $t$  is time, and  $q$  is the strength of the mass source distribution (monopole distribution, volume velocity distribution).

## Phi version

With an ideal gas, the equation of continuity can be written with the help of the state equation of the ideal gas (C.13) as

$$\frac{d\Pi}{dt} + \nabla \cdot \vec{U} = \frac{1}{c_p} \frac{dS}{dT} + q, \quad (\text{A.2})$$

where  $S$  is the entropy,  $T$  is the temperature,  $c_p$  is the specific heat at constant pressure and the pressure-density related field quantity  $\Pi$ , scaled logarithmic pressure, is defined in Eq. (C.12).

## Entropy version

From the energy equation (D.2), the following can be obtained

$$\frac{d\rho}{dt} = \frac{1}{c^2} \frac{dP}{dt} - \frac{\rho\beta T}{c_p} \frac{dS}{dt}, \quad (\text{A.3})$$

where  $P$  is the pressure,  $\beta$  is the coefficient of thermal expansion and  $c$  is the local speed of sound in constant entropy, defined by

$$c^2 = \left( \frac{\partial P}{\partial \rho} \right)_s, \quad (\text{A.4})$$

where  $()_f$  means that quantity  $f$  is constant in the operation inside the brackets.

By using this, the continuity equation (A.1) can be presented as

$$\frac{1}{\rho c^2} \frac{dP}{dt} + \nabla \cdot \vec{U} = \frac{\beta T}{c_p} \frac{dS}{dt} + q. \quad (\text{A.5})$$

## Enthalpy version

### Simple general fluids

The enthalpy difference  $dH$  can be presented as [10, 18]

$$dH = \frac{dP}{\rho} + T dS, \quad (\text{A.6})$$

where  $S$  is the entropy. The (specific) stagnation enthalpy  $B$  is defined by [10]

$$B = H + \frac{1}{2} \vec{U} \cdot \vec{U}. \quad (\text{A.7})$$

The gradient and the time derivative of the stagnation enthalpy, according to Eqs. (A.7) and (A.6), are

$$\begin{aligned} \nabla B &= \frac{1}{\rho} \nabla P + T \nabla S + \frac{1}{2} \nabla (\vec{U} \cdot \vec{U}) \\ \frac{dB}{dt} &= \frac{1}{\rho} \frac{dP}{dt} + T \frac{dS}{dt} + \vec{U} \cdot \frac{d\vec{U}}{dt}. \end{aligned} \quad (\text{A.8})$$

By using Eq. (A.8) (lower one), the continuity equation version (A.5) can be presented as

$$\frac{1}{c^2} \left( \frac{dB}{dt} - \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \nabla \cdot \vec{U} - \left( \frac{\beta T}{c_p} + \frac{T}{c^2} \right) \frac{dS}{dt} = q. \quad (\text{A.9})$$

### Ideal gas

Using Eqs. (C.9), (C.10), (C.8) and (C.3), the following can be obtained for an ideal gas

$$\frac{\beta T}{c_p} + \frac{T}{c^2} = \frac{1}{c_p} + \frac{\rho T}{\gamma P} = \frac{1}{c_p} + \frac{1}{\gamma R} = \frac{1}{c_p} + \frac{c_v}{c_p R} = \frac{R + c_v}{c_p R} = \frac{c_p}{c_p R} = \frac{1}{R}, \quad (\text{A.10})$$

where  $\gamma$  is the adiabatic constant,  $c_v$  is the specific heat at constant volume and  $R$  is the gas constant. The following equation for an ideal gas has also been used above [5]

$$c_p = c_v + R. \quad (\text{A.11})$$

Now the modified version (A.9) of the equation of continuity can be presented for an ideal gas as

$$\frac{1}{c^2} \left( \frac{dB}{dt} - \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \nabla \cdot \vec{U} - \frac{1}{R} \frac{dS}{dt} = q. \quad (\text{A.12})$$





# Appendix B: Navier–Stokes equation

## Basic version

The non-linear Navier–Stokes equation is [5 (without all source terms), 19 (with all source terms)]

$$\rho \frac{d\vec{U}}{dt} = \rho \left( \frac{\partial \vec{U}}{\partial t} + \vec{U} \cdot \nabla \vec{U} \right) = -\nabla P + \nabla \cdot \overset{\equiv}{\sigma}_\mu + \vec{F} - \nabla \cdot \overset{\equiv}{T}, \quad (\text{B.1})$$

where  $\rho$  is the density of the fluid,  $\vec{U}$  is the particle velocity,  $t$  is time,  $P$  is the pressure,  $\vec{F}$  is the strength of the force source distribution (dipole distribution + gravitation),  $\overset{\equiv}{T}$  is the strength of the momentum source distribution (quadrupole distribution), and  $\overset{\equiv}{\sigma}_\mu$  is the viscous part of the stress dyadic

$$\overset{\equiv}{\sigma}_\mu = \left( \mu_v \overset{\equiv}{E} : \overset{\equiv}{I} \right) \overset{\equiv}{I} + 2\mu \left[ \overset{\equiv}{E} - \frac{1}{3} \left( \overset{\equiv}{E} : \overset{\equiv}{I} \right) \overset{\equiv}{I} \right], \quad (\text{B.2})$$

where  $\mu$  is the coefficient of viscosity,  $\mu_v$  is the expansion coefficient of viscosity,  $\overset{\equiv}{I}$  is the identic dyadic, see Eq. (R.9), and  $\overset{\equiv}{E}$  is the rate-of-strain dyadic

$$\overset{\equiv}{E} = \frac{1}{2} \left[ \nabla \vec{U} + \left( \nabla \vec{U} \right)_T \right], \quad (\text{B.3})$$

where subscript ‘T’ denotes the transpose of a dyadic; see Eq. (R.11). For the dyadic notation as a whole, see Appendix R.

## Alternative version 1

By using the equation of continuity, Eq. (A.1), the Navier–Stokes equation can be presented as

$$\frac{\partial(\rho \vec{U})}{\partial t} + \nabla \cdot (\rho \vec{U} \vec{U}) = -\nabla P + \nabla \cdot \overset{\equiv}{\sigma}_\mu + \rho q \vec{U} + \vec{F} - \nabla \cdot \overset{\equiv}{T}, \quad (\text{B.4})$$

where  $q$  is the strength of the mass source distribution.

By adding and subtracting term  $\nabla(c^2\rho)$ , this can further be written as

$$\begin{aligned} \frac{\partial(\rho\vec{U})}{\partial t} + c^2\nabla\rho - \nabla \cdot \left[ \overline{\overline{\sigma}}_{\mu} - (P - c^2\rho)\overline{\overline{\mathbf{I}}} \right] + \rho\nabla c^2 \\ = -\nabla \cdot \left( \overline{\overline{T}} + \rho\vec{U}\vec{U} \right) + \rho q\vec{U} + \vec{F}, \end{aligned} \quad (\text{B.5})$$

where  $c$  is the local speed of sound in constant entropy, defined in Eq. (A.4).

## Alternative version 2

Using the following equation

$$\begin{aligned} \vec{U} \cdot (\nabla\vec{U}) &= \vec{U} \cdot (\nabla\vec{U}) + \vec{U} \times (\nabla \times \vec{U}) + (\nabla \times \vec{U}) \times \vec{U} \\ &= \frac{1}{2}\nabla(\vec{U} \cdot \vec{U}) + (\nabla \times \vec{U}) \times \vec{U} \\ &= \frac{1}{2}\nabla(\vec{U} \cdot \vec{U}) + \vec{\omega} \times \vec{U}, \end{aligned} \quad (\text{B.6})$$

(where, in the first line, it has been utilized that  $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$  for any vectors, and, in the second line, the upper line of Eq. (Q.5) has been used), the following can be obtained from Eq. (B.1)

$$\rho \left( \frac{\partial\vec{U}}{\partial t} + \frac{1}{2}\nabla(\vec{U} \cdot \vec{U}) + \vec{\omega} \times \vec{U} \right) = -\nabla P + \nabla \cdot \overline{\overline{\sigma}}_{\mu} + \vec{F} - \nabla \cdot \overline{\overline{T}}, \quad (\text{B.7})$$

where  $\vec{\omega}$  is the vorticity distribution

$$\vec{\omega} = \nabla \times \vec{U}. \quad (\text{B.8})$$

## Phi version

With an ideal gas, the Navier-Stokes equation (B.1) can be written with the help of the state equation of the ideal gas (C.13) as

$$\frac{d\vec{U}}{dt} = -c^2 \nabla \Pi + \frac{1}{\rho} \left( \nabla \cdot \overset{=}{\sigma}_\mu + \vec{F} - \nabla \cdot \overset{=}{T} \right), \quad (\text{B.9})$$

where the field quantity  $\Pi$ , scaled logarithmic pressure, is defined in Eq. (C.12) and where, according to Eqs. (C.12) and (C.10) in ideal gas, we have connection

$$c^2 \nabla \Pi = \frac{1}{\rho} \nabla P. \quad (\text{B.10})$$

## Enthalpy version

Using definitions (A.7) and (A.6) for stagnation enthalpy  $B$ , the Navier–Stokes equation version (B.7) can be presented as

$$\frac{\partial \vec{U}}{\partial t} + \nabla B = -\vec{\omega} \times \vec{U} + T \nabla S + \frac{1}{\rho} \left( \nabla \cdot \overset{=}{\sigma}_\mu + \vec{F} - \nabla \cdot \overset{=}{T} \right), \quad (\text{B.11})$$

where Eq. (A.8) (upper one) has also been used.

Let us define vector  $\vec{V}$  by the equation

$$\vec{V} = T \nabla S + \frac{1}{\rho} \left( \nabla \cdot \overset{=}{\sigma}_\mu + \vec{F} - \nabla \cdot \overset{=}{T} \right). \quad (\text{B.12})$$

The Navier–Stokes equation (B.11) can now be written as

$$\frac{\partial \vec{U}}{\partial t} = -\nabla B - \vec{\omega} \times \vec{U} + \vec{V}. \quad (\text{B.13})$$



# Appendix C: State equation

## Simple general fluids

For simple fluids, the state equation can be expressed so that the number of independent properties needed to specify the state of the fluid is two. It is immaterial which two, and if two properties are specified, then all other properties have fixed values. [18]

From this origin, the state equation can be expressed as, e.g.,

$$P = P(\rho, S), \quad (\text{C.1})$$

where  $P$  is pressure,  $\rho$  is density and  $S$  is entropy.

From this equation, a general state equation can immediately be written

$$dP = \left( \frac{\partial P}{\partial \rho} \right)_S d\rho + \left( \frac{\partial P}{\partial S} \right)_\rho dS, \quad (\text{C.2})$$

where  $(\ )_f$  means that quantity  $f$  is constant in the operation inside the brackets.

By using the following equations and definitions [18]

$$\begin{aligned} T dS &= c_v dT + T \left( \frac{\partial P}{\partial T} \right)_{V^*} dV^* \\ T dS &= c_p dT - \beta T V^* dP \\ \left( \frac{\partial P}{\partial S} \right)_\rho &= - \left( \frac{\partial T}{\partial V^*} \right)_S \quad \left( \frac{\partial P}{\partial T} \right)_{V^*} = \left( \frac{\partial S}{\partial V^*} \right)_T \\ \beta &= \frac{1}{V^*} \left( \frac{\partial V^*}{\partial T} \right)_P \\ c^2 &= \left( \frac{\partial P}{\partial \rho} \right)_S \quad c_T^2 = \left( \frac{\partial P}{\partial \rho} \right)_T = \frac{c^2}{\gamma} \\ \gamma &= \frac{c_p}{c_v} \quad V^* = \frac{1}{\rho}, \end{aligned} \quad (\text{C.3})$$

where  $T$  is the temperature,  $V^*$  is the specific volume,  $c_P$  and  $c_V$  are the specific heats at constant pressure and volume respectively,  $\gamma$  is the adiabatic constant,  $\beta$  is the coefficient of thermal expansion,  $c$  is the local (isentropic) speed of sound and  $c_T$  is the isothermal speed of sound, it is clear that the second partial derivative on the right-hand side of Eq. (C.2) is

$$\left(\frac{\partial P}{\partial S}\right)_\rho = \frac{\rho c^2 \beta T}{c_P}. \quad (\text{C.4})$$

The first partial derivative on the right-hand side of Eq. (C.2) can be identified to be the local speed of sound squared. With the help of this and Eq. (C.4), the state equation (C.2) can be written more specifically as

$$dP = c^2 d\rho + \frac{\rho c^2 \beta T}{c_P} dS. \quad (\text{C.5})$$

From the equation above, the entropy change can be written as

$$dS = \frac{c_P}{\rho c^2 \beta T} (dP - c^2 d\rho) \quad (\text{C.6})$$

and the proportional density change as

$$\frac{d\rho}{\rho} = \frac{dP}{\rho c^2} - \frac{\beta T}{c_P} dS. \quad (\text{C.7})$$

## Ideal gas

With an ideal gas

$$PV^* = RT, \quad (\text{C.8})$$

where  $R$  is the gas constant, so the coefficient of thermal expansion for an ideal gas is

$$\beta = \frac{1}{V^*} \left( \frac{\partial V^*}{\partial T} \right)_p = \frac{1}{V^*} \frac{R}{P} = \frac{1}{T}. \quad (\text{C.9})$$

The local speed of sound for an ideal gas can be written, according to Eqs. (C.3), (C.8) and (C.9), as

$$c^2 = \left( \frac{\partial P}{\partial \rho} \right)_s = -V^{*2} \left( \frac{\partial P}{\partial V^*} \right)_s = \frac{\gamma P}{\rho}. \quad (\text{C.10})$$

Now the state equation (C.7) can be written for an ideal gas using Eqs. (C.9) and (C.10), as

$$\frac{d\rho}{\rho} = \frac{dP}{\gamma P} - \frac{dS}{c_p}. \quad (\text{C.11})$$

By defining a new field quantity  $\Pi$ , scaled logarithmic pressure, as

$$\Pi = \frac{1}{\gamma} \ln \left( \frac{P}{P_0} \right), \quad (\text{C.12})$$

where  $P_0$  is some convenient constant reference pressure, the state equation for an ideal gas (C.11) can be written as

$$\frac{d\rho}{\rho} = d\Pi - \frac{dS}{c_p}. \quad (\text{C.13})$$





# Appendix D: Energy equation

## Entropy version

The energy equation for a Newtonian fluid is [18 (without source term), 19 (with source term)]

$$\begin{aligned} T \frac{dS}{dt} &= T \left( \frac{\partial S}{\partial t} + \vec{U} \cdot \nabla S \right) \\ &= \frac{1}{\rho} \left( \overline{\overline{\sigma}}_{\mu} : \overline{\overline{E}} + \nabla \cdot (K \nabla T) + \frac{d\varepsilon}{dt} \right), \end{aligned} \quad (\text{D.1})$$

where  $T$  is the temperature,  $S$  is the entropy,  $t$  is the time,  $\vec{U}$  is the particle velocity,  $\underline{\underline{\rho}}$  is the density,  $\overline{\overline{\sigma}}_{\mu}$  is the viscous part of the stress dyadic defined in Eq. (B.2),  $\overline{\overline{E}}$  is the rate-of-strain dyadic defined in Eq. (B.3),  $K$  is the thermal conductivity of the fluid and  $\varepsilon$  is the energy per unit volume delivered by the heat source distribution. With the help of Eq. (C.6), the right-hand side of Eq. (D.1) is the same as

$$\begin{aligned} T \frac{dS}{dt} &= T \left( \frac{\partial S}{\partial t} + \vec{U} \cdot \nabla S \right) \\ &= \frac{c_P}{\rho c^2 \beta} \left( \frac{dP}{dt} - c^2 \frac{d\rho}{dt} \right) \\ &= \frac{c_P}{\rho c^2 \beta} \left[ \frac{\partial P}{\partial t} + \vec{U} \cdot \nabla P - c^2 \left( \frac{\partial \rho}{\partial t} + \vec{U} \cdot \nabla \rho \right) \right]. \end{aligned} \quad (\text{D.2})$$

where  $P$  is the pressure,  $c_P$  is the specific heat at constant pressure,  $\beta$  is the coefficient of thermal expansion and  $c$  is the local (isentropic) speed of sound.

## Internal energy version

The energy balance equation can also be presented in another way. The rate of change of the internal energy  $E_{\text{int}}$  per unit mass is [18 (without source terms), 19 (with source terms)]

$$\frac{dE_{\text{int}}}{dt} = \frac{1}{\rho} \left[ -P \nabla \cdot \vec{U} + \overline{\overline{\sigma}}_{\mu} : \overline{\overline{E}} + \nabla \cdot (K \nabla T) + qP + \frac{d\varepsilon}{dt} \right], \quad (\text{D.3})$$

where  $q$  is the strength of the mass source distribution.

By forming a dot product of the Navier-Stokes equation (B.1) with the particle velocity we can obtain

$$\rho \vec{U} \cdot \frac{d\vec{U}}{dt} = -\nabla P \cdot \vec{U} + \left( \nabla \cdot \overline{\overline{\sigma}}_{\mu} \right) \cdot \vec{U} + \vec{F} \cdot \vec{U} - \left( \nabla \cdot \overline{\overline{T}} \right) \cdot \vec{U}, \quad (\text{D.4})$$

where  $\vec{F}$  is the strength of the force source distribution and  $\overline{\overline{T}}$  is the momentum source distribution.

Using Eqs. (D.3) and (D.4) we obtain

$$\begin{aligned} & \nabla \cdot (P \vec{U} - \overline{\overline{\sigma}}_{\mu} \cdot \vec{U} - K \nabla T) \\ &= \nabla P \cdot \vec{U} + P \nabla \cdot \vec{U} - \left( \nabla \cdot \overline{\overline{\sigma}}_{\mu} \right) \cdot \vec{U} - \overline{\overline{\sigma}}_{\mu} : \overline{\overline{E}} - \nabla \cdot (K \nabla T) \\ &= -\rho \frac{d}{dt} \left( E_{\text{int}} + \frac{1}{2} \vec{U} \cdot \vec{U} \right) + qP + \vec{F} \cdot \vec{U} - \left( \nabla \cdot \overline{\overline{T}} \right) \cdot \vec{U} + \frac{d\varepsilon}{dt}, \end{aligned} \quad (\text{D.5})$$

where, due to the symmetry of the viscous part of the stress dyadic and the definition of the rate-of-strain dyadic (B.3), we have, using Eq. (R.16) (eighth line),

$$\begin{aligned} \nabla \cdot \left( \overline{\overline{\sigma}}_{\mu} \cdot \vec{U} \right) &= \left( \nabla \cdot \overline{\overline{\sigma}}_{\mu} \right) \cdot \vec{U} + \overline{\overline{\sigma}}_{\mu} : \nabla \vec{U} + \left( \nabla \cdot \overline{\overline{\sigma}}_{\mu} \right) \cdot \vec{U} + \overline{\overline{\sigma}}_{\mu} : \frac{1}{2} \left[ \nabla \vec{U} + \left( \nabla \vec{U} \right)_{\text{T}} \right] \\ &= \left( \nabla \cdot \overline{\overline{\sigma}}_{\mu} \right) \cdot \vec{U} + \overline{\overline{\sigma}}_{\mu} : \overline{\overline{E}}, \end{aligned} \quad (\text{D.6})$$

where subscript ‘T’ denotes the transpose of a dyadic; see Eq. (R.11).

Using the equation of continuity (A.1), Eq. (D.5) can be presented as

$$\begin{aligned}
& \nabla \cdot (P\vec{U} - \overline{\overline{\sigma}}_{\mu} \cdot \vec{U} - K\nabla T) = \\
& = -\frac{d}{dt}(\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U}) + \frac{d\rho}{dt}(E_{\text{int}} + \frac{1}{2}\vec{U} \cdot \vec{U}) + \\
& \quad + qP + \vec{F} \cdot \vec{U} - \left(\nabla \cdot \overline{\overline{T}}\right) \cdot \vec{U} + \frac{d\varepsilon}{dt} = \\
& = -\frac{d}{dt}(\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U}) - (\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U})\nabla \cdot \vec{U} + \\
& \quad + (\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U})q + qP + \vec{F} \cdot \vec{U} - \left(\nabla \cdot \overline{\overline{T}}\right) \cdot \vec{U} + \frac{d\varepsilon}{dt} = \\
& = -\frac{\partial}{\partial t}(\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U}) - \nabla \cdot [(\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U})\vec{U}] + \\
& \quad + (\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U})q + qP + \vec{F} \cdot \vec{U} - \left(\nabla \cdot \overline{\overline{T}}\right) \cdot \vec{U} + \frac{d\varepsilon}{dt}.
\end{aligned} \tag{D.7}$$

This can be further presented as

$$\begin{aligned}
& \frac{\partial}{\partial t}(\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U}) + \nabla \cdot [(\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U})\vec{U}] + \\
& \quad + \nabla \cdot (P\vec{U} - \overline{\overline{\sigma}}_{\mu} \cdot \vec{U} - K\nabla T) = \\
& = (\rho E_{\text{int}} + \frac{1}{2}\rho\vec{U} \cdot \vec{U})q + Pq + \vec{F} \cdot \vec{U} - \left(\nabla \cdot \overline{\overline{T}}\right) \cdot \vec{U} + \frac{d\varepsilon}{dt}.
\end{aligned} \tag{D.8}$$

The final form (D.8) has been presented, e.g., in Ref. [8], without the mass and momentum source distributions.



## Appendix E: Linearization process

Next, some equations will be partly linearized. The linearization is necessary to obtain the proper relationships for the perturbation fields, but full linearization would lead to a situation in which part of the sound source terms that we are looking for would be lost in the linearization. This presentation is based on the basic version of the continuity equation (A.1), the alternative version 1 of the Navier–Stokes equation (B.5) and the entropy versions of the energy equations (D.1) and (D.2). Other versions of these equations could also be selected, but this selection is to support the derivation of Lighthill’s equation in Appendix H.

Let us suppose that the field quantities  $P$ ,  $\rho$ ,  $T$  and  $S$  can be expressed by the sum of static components  $P_0$ ,  $\rho_0$ ,  $T_0$  and  $S_0$ , and perturbation components  $p$ ,  $\rho'$ ,  $T'$  and  $s$  in such a way that the perturbation components are much smaller than the static ones

$$\begin{aligned}
 P(\vec{r}, t) &= P_0(\vec{r}) + p(\vec{r}, t) \\
 \rho(\vec{r}, t) &= \rho_0(\vec{r}) + \rho'(\vec{r}, t) \\
 T(\vec{r}, t) &= T_0(\vec{r}) + T'(\vec{r}, t) \\
 S(\vec{r}, t) &= S_0(\vec{r}) + s(\vec{r}, t) \\
 |p(\vec{r}, t)| &\ll P_0(\vec{r}) \\
 |\rho'(\vec{r}, t)| &\ll \rho_0(\vec{r}) \\
 |T'(\vec{r}, t)| &\ll T_0(\vec{r}) \\
 |s(\vec{r}, t)| &\ll S_0(\vec{r}),
 \end{aligned} \tag{E.1}$$

and that the time averages of the perturbation components are zero. The perturbation components are functions of time  $t$  and the spatial coordinates  $\vec{r}$ , while the static components are only functions of spatial coordinates.

Let us also suppose that the viscous part of the stress dyadic, the rate-of-strain dyadic, and the particle velocity can be expressed as sums of a component related to the static fields (subscript ‘0’) and a component related to the perturbation fields

$$\begin{aligned}
 \overline{\overline{\sigma}}_{\mu}(\vec{r}, t) &= \overline{\overline{\sigma}}_{\mu 0}(\vec{r}) + \overline{\overline{\sigma}}'_{\mu}(\vec{r}, t) \\
 \overline{\overline{E}}(\vec{r}, t) &= \overline{\overline{E}}_0(\vec{r}) + \overline{\overline{e}}(\vec{r}, t) \\
 \overline{\overline{U}}(\vec{r}, t) &= \overline{\overline{U}}_0(\vec{r}) + \overline{\overline{u}}(\vec{r}, t) \\
 \overline{\overline{\sigma}}_{\mu 0}(\vec{r}) &= \left( \overline{\overline{\mu}}_{\nu} \overline{\overline{E}}_0 : \overline{\overline{I}} \right) \overline{\overline{I}} + 2\mu \left[ \overline{\overline{E}}_0 - \frac{1}{3} \left( \overline{\overline{E}}_0 : \overline{\overline{I}} \right) \overline{\overline{I}} \right] \\
 \overline{\overline{\sigma}}'_{\mu}(\vec{r}, t) &= \left( \overline{\overline{\mu}}_{\nu} \overline{\overline{e}} : \overline{\overline{I}} \right) \overline{\overline{I}} + 2\mu \left[ \overline{\overline{e}} - \frac{1}{3} \left( \overline{\overline{e}} : \overline{\overline{I}} \right) \overline{\overline{I}} \right] \\
 \overline{\overline{E}}_0 &= \frac{1}{2} \left[ \nabla \overline{\overline{U}}_0 + (\nabla \overline{\overline{U}}_0)_{\text{T}} \right] \\
 \overline{\overline{e}} &= \frac{1}{2} \left[ \nabla \overline{\overline{u}} + (\nabla \overline{\overline{u}})_{\text{T}} \right].
 \end{aligned} \tag{E.2}$$

Let us further suppose that the monopole, quadrupole and heat source distributions  $q$ ,  $T$  and  $\varepsilon$  are connected to the perturbation fields and that force source distribution can be divided into a static gravitational force and a perturbation force  $\vec{f}$

$$\vec{F}(\vec{r}, t) = \rho_0(\vec{r}) \vec{g}(\vec{r}) + \vec{f}(\vec{r}, t), \tag{E.3}$$

where  $\vec{g}$  is the acceleration of gravity.

In the absence of the perturbation components, the static components obey, e.g., static parts of Eqs. (A.1), (B.5), (D.1) and (D.2), which are

$$\begin{aligned}
 \nabla \cdot (\rho_0 \vec{U}_0) &= 0 \\
 -\nabla \cdot \left( \overline{\overline{\sigma}}_{\mu 0} - P_0 \overline{\overline{I}} - \rho_0 \vec{U}_0 \vec{U}_0 \right) &= \rho_0 \vec{g} \\
 \rho_0 T_0 \vec{U}_0 \cdot \nabla S_0 = \overline{\overline{\sigma}}_{\mu 0} : \overline{\overline{E}}_0 + \nabla \cdot (K \nabla T_0) &= \frac{c_P}{c^2 \beta} \left( \vec{U}_0 \cdot \nabla P_0 - c^2 \vec{U}_0 \cdot \nabla \rho_0 \right).
 \end{aligned} \tag{E.4}$$

It is supposed that with the perturbation fields the static fields obey the same equations so that the perturbation fields have no effect on the static ones. In this case, the equations of the static fields can be extracted from the total field equations to obtain the equations for the perturbation fields. The energy equation (D.2) for fields in which static fields are extracted is

$$\begin{aligned}
& \rho T \left( \frac{\partial s}{\partial t} + \vec{U} \cdot \nabla S \right) - \rho_0 T_0 \vec{U}_0 \cdot \nabla S_0 \\
&= \frac{c_P}{c^2 \beta} \left[ \frac{\partial p}{\partial t} + \vec{U} \cdot \nabla P - \vec{U}_0 \cdot \nabla P_0 - c^2 \left( \frac{\partial \rho'}{\partial t} + \vec{U} \cdot \nabla \rho - \vec{U}_0 \cdot \nabla \rho_0 \right) \right],
\end{aligned} \tag{E.5}$$

which can be rearranged into the form

$$\begin{aligned}
& \rho T \left( \frac{\partial s}{\partial t} + \vec{U} \cdot \nabla s \right) + (\rho T \vec{U} - \rho_0 T_0 \vec{U}_0) \cdot \nabla S_0 \\
&= \frac{c_P}{c^2 \beta} \left[ \frac{\partial p}{\partial t} + \vec{U} \cdot \nabla p + \vec{u} \cdot \nabla P_0 - c^2 \left( \frac{\partial \rho'}{\partial t} + \vec{U} \cdot \nabla \rho' + \vec{u} \cdot \nabla \rho_0 \right) \right] \\
&\Leftrightarrow \\
& \rho T \frac{ds}{dt} + (\rho T \vec{U} - \rho_0 T_0 \vec{U}_0) \cdot \nabla S_0 \\
&= \frac{c_P}{c^2 \beta} \left[ \frac{dp}{dt} - c^2 \frac{d\rho'}{dt} + \vec{u} \cdot (\nabla P_0 - c^2 \cdot \nabla \rho_0) \right].
\end{aligned} \tag{E.6}$$

Suppose that the gradients of the static field quantities  $P_0$ ,  $\rho_0$  and  $S_0$  are small, at most of the perturbation order, and that the local speed of sound is not a function of time. In this case, the second version of Eq. (E.6) can be linearized by eliminating all second and higher order terms that have these gradients as part of them to yield

$$\rho T \frac{ds}{dt} = \frac{c_P}{c^2 \beta} \left( \frac{dp}{dt} - c^2 \frac{d\rho'}{dt} \right) = \frac{c_P}{c^2 \beta} \frac{d}{dt} (p - c^2 \rho'). \tag{E.7}$$

Now, it can be seen that the linearized presentation for the entropy perturbation is

$$s = \frac{c_P}{\rho c^2 \beta T} (p - c^2 \rho') \tag{E.8}$$

and its time derivative, according to Eqs. (D.1), (E.2) and (E.4), is

$$\begin{aligned}\frac{ds}{dt} &= \left( \frac{\partial s}{\partial t} + \vec{U} \cdot \nabla s \right) \\ &= \frac{1}{\rho T} \left( \overline{\overline{\sigma}}_{\mu} : \overline{\overline{E}} - \overline{\overline{\sigma}}_{\mu 0} : \overline{\overline{E}}_0 + \nabla \cdot (K \nabla T') + \frac{d\varepsilon}{dt} \right).\end{aligned}\tag{E.9}$$

The equation of continuity version (A.1) and the Navier–Stokes equation version (B.5) with the static fields extracted are

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho \vec{U} - \rho_0 \vec{U}_0) = \rho q\tag{E.10}$$

$$\begin{aligned}\vec{U} \frac{\partial \rho'}{\partial t} + \rho \frac{\partial \vec{u}}{\partial t} + c^2 \nabla \rho' - \nabla \cdot \left[ \overline{\overline{\sigma}}'_{\mu} - (p - c^2 \rho') \overline{\overline{I}} \right] + \rho' \nabla c^2 = \\ = -\nabla \cdot \left( \overline{\overline{T}} + \rho \vec{U} \vec{U} - \rho_0 \vec{U}_0 \vec{U}_0 \right) + \rho q \vec{U} + \vec{f}.\end{aligned}\tag{E.11}$$

Suppose the gradient of  $c^2$  is small, at most of the perturbation order. Using this and Eq. (E.8), the Navier–Stokes equation (E.11) with the static fields extracted can be presented as

$$\begin{aligned}\vec{U} \frac{\partial \rho'}{\partial t} + \rho \frac{\partial \vec{u}}{\partial t} + c^2 \nabla \rho' - \nabla \cdot \left( \overline{\overline{\sigma}}'_{\mu} - \frac{\rho c^2 \beta T}{c_p} s \overline{\overline{I}} \right) \\ = -\nabla \cdot \left( \overline{\overline{T}} + \rho \vec{U} \vec{U} - \rho_0 \vec{U}_0 \vec{U}_0 \right) + \rho q \vec{U} + \vec{f}.\end{aligned}\tag{E.12}$$



## Appendix F: Sound radiation from source distributions

In the basic equations presented in the previous appendices, three types of true sound sources have been introduced:  $q$  as the monopole distribution (mass source distribution, volume velocity distribution),  $\vec{f}$  as the dipole distribution (force source distribution) and  $\overline{\overline{T}}$  as the quadrupole distribution (momentum source distribution). The heat source distribution is assumed to be included in the monopole distribution. After linearization of the field equations, see Appendix E, the sound radiation of each, presented here as  $p_q$ ,  $p_f$  and  $p_T$ , and their total sound radiation, presented as  $p$ , can be expressed as [5, 12, 19]

$$\begin{aligned}
 p(\vec{r}, t) &= p_q(\vec{r}, t) + p_f(\vec{r}, t) + p_T(\vec{r}, t) \\
 p_q(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_V \rho_0 \frac{\partial q(\vec{r}_0, t_0)}{\partial t_0} g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 p_f(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_V \vec{f}(\vec{r}_0, t_0) \cdot \nabla_0 g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 p_T(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_V \overline{\overline{T}}(\vec{r}_0, t_0) : \nabla_0 \nabla_0 g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0,
 \end{aligned} \tag{F.1}$$

where  $\vec{r}$  is a field point vector,  $\vec{r}_0$  is a source point vector,  $t$  is time,  $t_0$  is the ‘source time variable’,  $V$  is the volume containing the source distributions, gradient  $\nabla$  operates on the field coordinates, gradient  $\nabla_0$  operates on the source coordinates, and  $g$  is Green’s function. The time integration is applied to variable  $t_0$  and the volume integration is applied to coordinates of  $\vec{r}_0$ . Green’s function for free space  $g_0$  is

$$g_0(\vec{r}, t | \vec{r}_0, t_0) = \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|}, \tag{F.2}$$

where  $c$  is the speed of sound and  $\delta$  is the Dirac delta function.

The field of the monopole distribution can be presented in another form. It is easy to notice by partial integration that

$$\begin{aligned}
 p_q(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_V \rho_0 \frac{\partial q(\vec{r}_0, t_0)}{\partial t_0} g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 &= \int_{t_0=-\infty}^{t+\Delta t} \int_V \rho_0 \frac{\partial q(\vec{r}_0, t_0)}{\partial t_0} g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 &= \int_{t_0=-\infty}^{t+\Delta t} \int_V \rho_0 \frac{\partial}{\partial t_0} [q(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0)] dV dt_0 \\
 &\quad - \int_{t_0=-\infty}^{t+\Delta t} \int_V \rho_0 q(\vec{r}_0, t_0) \frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t_0} dV dt_0 \\
 &= - \int_{t_0=-\infty}^{t+\Delta t} \int_V \rho_0 q(\vec{r}_0, t_0) \frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t_0} dV dt_0 \\
 &= - \int_{t_0=-\infty}^t \int_V \rho_0 q(\vec{r}_0, t_0) \frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t_0} dV dt_0.
 \end{aligned} \tag{F.3}$$

An extension of the upper integration bound to  $t + \Delta t$  (and the opposite action back) can be made because the causality of Green's function demands Green's function to be zero if  $t_0 > t$ . The first term in the third version on the right-hand side of the equation above disappears because the subsequent time derivation and integration lead to the difference in the function values inside the square brackets at the integration path end points where the function values are zero (if Green's function can be thought to vanish also below some lower time bound).

If Green's function is reciprocal like the free space Green's function (F.2), we have

$$\begin{aligned}
 \nabla_0 g(\vec{r}, t | \vec{r}_0, t_0) &= -\nabla g(\vec{r}, t | \vec{r}_0, t_0) \\
 \frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t_0} &= -\frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t}.
 \end{aligned} \tag{F.4}$$

In this case, changing  $\nabla_0$  to  $-\nabla$ , the gradient  $\nabla$  can be applied as a divergence to the whole integrand in  $p_f$  and  $p_T$  (source distributions are not functions of  $\vec{r}$ ) and the divergence operator(s) can furthermore put outside the integrals. Similarly, changing  $t_0$  to  $t$  in the time derivation, the time derivation can be applied to the whole integrand in  $p_q$  (source distributions are not functions of  $t$ ) and the time derivation can furthermore put outside the integral. Thus, in this case, the sound

radiation from the source distributions can be presented, according to Eqs. (F.1), (F.3) and (F.4), in an alternative form as

$$\begin{aligned}
 p_q(\vec{r}, t) &= \frac{\partial}{\partial t} \int_{t_0=-\infty}^t \int_V \rho_0 q(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 p_f(\vec{r}, t) &= -\nabla \cdot \int_{t_0=-\infty}^t \int_V \vec{f}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 p_T(\vec{r}, t) &= \nabla \nabla : \int_{t_0=-\infty}^t \int_V \vec{T}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 .
 \end{aligned} \tag{F.5}$$

Thus, the sound radiation from the source distributions can be presented as

$$p(\vec{r}, t) = p_q(\vec{r}, t) + p_f(\vec{r}, t) + p_T(\vec{r}, t), \tag{F.6}$$

where the monopole, dipole and quadrupole parts can be presented in the alternative forms

$$\begin{aligned}
 p_q(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_V \rho_0 \frac{\partial q(\vec{r}_0, t_0)}{\partial t_0} g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 &= - \int_{t_0=-\infty}^t \int_V \rho_0 q(\vec{r}_0, t_0) \frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t_0} dV dt_0 \\
 &= \frac{\partial}{\partial t} \int_{t_0=-\infty}^t \int_V \rho_0 q(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0
 \end{aligned} \tag{F.7}$$

$$\begin{aligned}
 p_f(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_V \vec{f}(\vec{r}_0, t_0) \cdot \nabla_0 g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 &= -\nabla \cdot \int_{t_0=-\infty}^t \int_V \vec{f}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0
 \end{aligned} \tag{F.8}$$

$$\begin{aligned}
 p_T(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_V \overline{\overline{T}}(\vec{r}_0, t_0) : \nabla_0 \nabla_0 g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \\
 &= \nabla \nabla : \int_{t_0=-\infty}^t \int_V \overline{\overline{T}}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0,
 \end{aligned} \tag{F.9}$$

where the last alternatives suppose that Green's function is reciprocal.

If the source distributions are surface distributions  $q_s$ ,  $\vec{f}_s$  and  $T_s$  instead of volume distributions

$$\begin{aligned}
 q &= q_s \delta(w_1 - w_{10}) \\
 \vec{f} &= \vec{f}_s \delta(w_1 - w_{10}) \\
 \overline{\overline{T}} &= \overline{\overline{T}}_s \delta(w_1 - w_{10}),
 \end{aligned} \tag{F.10}$$

where the distributions are on the surface  $S$  with  $w_1 = w_{10}$  in a coordinate system  $(w_1, w_2, w_3)$ , we obtain

$$\begin{aligned}
 p_q(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_S \rho_0 \frac{\partial q_s(\vec{r}_0, t_0)}{\partial t_0} g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0 \\
 &= - \int_{t_0=-\infty}^t \int_S \rho_0 q_s(\vec{r}_0, t_0) \frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t_0} dS dt_0 \\
 &= \frac{\partial}{\partial t} \int_{t_0=-\infty}^t \int_S \rho_0 q_s(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0
 \end{aligned} \tag{F.11}$$

$$\begin{aligned}
 p_f(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_S \vec{f}_s(\vec{r}_0, t_0) \cdot \nabla_0 g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0 \\
 &= -\nabla \cdot \int_{t_0=-\infty}^t \int_S \vec{f}_s(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0
 \end{aligned} \tag{F.12}$$

$$\begin{aligned}
 p_T(\vec{r}, t) &= \int_{t_0=-\infty}^t \int_S \overline{\overline{T}}_s(\vec{r}_0, t_0) : \nabla_0 \nabla_0 g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0 \\
 &= \nabla \nabla : \int_{t_0=-\infty}^t \int_S \overline{\overline{T}}_s(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0.
 \end{aligned} \tag{F.13}$$

If the free space Green's function (F.2) is applied above, the sound radiation from the volume source distributions can be presented as

$$\begin{aligned}
 p_q(\vec{r}, t) &= \int_V \rho_0 \frac{\partial q(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c)}{\partial t} \frac{1}{4\pi|\vec{r} - \vec{r}_0|} dV \\
 p_f(\vec{r}, t) &= -\nabla \cdot \int_V \vec{f}(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c) \frac{1}{4\pi|\vec{r} - \vec{r}_0|} dV \\
 p_T(\vec{r}, t) &= \nabla \nabla : \int_V \vec{T}(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c) \frac{1}{4\pi|\vec{r} - \vec{r}_0|} dV
 \end{aligned} \tag{F.14}$$

and from the surface source distributions as

$$\begin{aligned}
 p_q(\vec{r}, t) &= \int_S \rho_0 \frac{\partial q_s(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c)}{\partial t} \frac{1}{4\pi|\vec{r} - \vec{r}_0|} dS \\
 p_f(\vec{r}, t) &= -\nabla \cdot \int_S \vec{f}_s(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c) \frac{1}{4\pi|\vec{r} - \vec{r}_0|} dS \\
 p_T(\vec{r}, t) &= \nabla \nabla : \int_S \vec{T}_s(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c) \frac{1}{4\pi|\vec{r} - \vec{r}_0|} dS.
 \end{aligned} \tag{F.15}$$



# Appendix G: Equivalent Huygens sources

## Surface field discontinuities as sources

Next, the quadrupole source distribution is assumed to vanish and the monopole and dipole source distributions are supposed to form a planar surface situated at plane  $x = 0$ , i.e.,

$$\begin{aligned} q &= q_s \delta(x) \\ \vec{F} &= \vec{F}_s \delta(x), \end{aligned} \tag{G.1}$$

where  $\delta(x)$  is a Dirac delta function. Quantities  $q_s$  and  $\vec{F}_s$  are thus planar source densities (volume velocity and force per unit area). Inserting these into the basic version of the continuity equation (A.1) and the basic version of the Navier–Stokes equation (B.1) shows that the Dirac delta function can only be obtained from the discontinuity of the divergence and gradient terms in the equations. By inserting the expression

$$\nabla \cdot (\rho \vec{U}) = (\nabla \rho) \cdot \vec{U} + \rho \nabla \cdot \vec{U} \tag{G.2}$$

into the equation of continuity (A.1), it can be noted that the density must contain the term  $\Delta \rho H(x)$  and the velocity must contain the term  $\Delta \vec{U} H(x)$ , where  $H(x)$  is a step function (Heaviside function) according to Eq. (14). Taking the gradient of the density and the divergence of the velocity yields Dirac delta functions from the step functions. Integrating the continuity equation over a small path across  $x = 0$ , the planar monopole distribution can be seen to be

$$q_s = \left( \Delta \vec{U} + \frac{\Delta \rho}{\rho} \vec{U} \right) \cdot \vec{e}_x, \tag{G.3}$$

where  $\vec{e}_x$  is a unit vector in the  $x$  direction.

Using the Navier–Stokes equation (B.1), it can be noted that to obtain Dirac delta functions there, the pressure must contain the term  $\Delta P H(x)$ , the velocity must contain the term  $\Delta \vec{U} H(x)$  and the viscous part of the stress dyadic must

contain the term  $\overline{\Delta\sigma_\mu}H(x)$ . Inserting all the discontinuous terms into the Navier–Stokes equation (B.1) and integrating it over a small path across  $x = 0$ , the planar dipole distribution can be seen to be

$$\vec{F}_s = \Delta P \vec{e}_x + \rho \vec{U} \cdot \vec{e}_x \Delta \vec{U} - \vec{e}_x \cdot \overline{\Delta\sigma_\mu}. \quad (\text{G.4})$$

Thus, the planar source distributions can be obtained from the field discontinuities using Eqs. (G.3) and (G.4).

If the source distribution surface is not planar, it has to be handled in general curvilinear coordinates. Let  $(w_1, w_2, w_3)$  form a curvilinear coordinate system such that the location of the source distribution can be presented with a constant  $w_1$  surface  $w_1 = w_{10}$ , where  $w_{10}$  is constant. In this case, the source distributions are as in Eq. (G.1), with  $\delta(x)$  replaced by  $\delta(w_1 - w_{10})$

$$\begin{aligned} q &= q_s \delta(w_1 - w_{10}) \\ \vec{F} &= \vec{F}_s \delta(w_1 - w_{10}). \end{aligned} \quad (\text{G.5})$$

In curvilinear coordinates we have

$$\begin{aligned} \nabla a &= \frac{1}{h_1} \frac{\partial a}{\partial w_1} + \frac{1}{h_2} \frac{\partial a}{\partial w_2} + \frac{1}{h_3} \frac{\partial a}{\partial w_3} \\ \nabla \vec{a} &= \vec{e}_1 \frac{1}{h_1} \frac{\partial \vec{a}}{\partial w_1} + \vec{e}_2 \frac{1}{h_2} \frac{\partial \vec{a}}{\partial w_2} + \vec{e}_3 \frac{1}{h_3} \frac{\partial \vec{a}}{\partial w_3} \\ \nabla \cdot \vec{a} &= \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial w_1} (h_2 h_3 \vec{a} \cdot \vec{e}_1) + \frac{\partial}{\partial w_2} (h_3 h_1 \vec{a} \cdot \vec{e}_2) + \frac{\partial}{\partial w_3} (h_1 h_2 \vec{a} \cdot \vec{e}_3) \right] \\ \nabla \times \vec{a} &= \frac{1}{h_2 h_3} \left[ \frac{\partial}{\partial w_2} (h_3 \vec{a} \cdot \vec{e}_3) - \frac{\partial}{\partial w_3} (h_2 \vec{a} \cdot \vec{e}_2) \right] \vec{e}_1 \\ &\quad + \frac{1}{h_3 h_1} \left[ \frac{\partial}{\partial w_3} (h_1 \vec{a} \cdot \vec{e}_1) - \frac{\partial}{\partial w_1} (h_3 \vec{a} \cdot \vec{e}_3) \right] \vec{e}_2 \\ &\quad + \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial w_1} (h_2 \vec{a} \cdot \vec{e}_2) - \frac{\partial}{\partial w_2} (h_1 \vec{a} \cdot \vec{e}_1) \right] \vec{e}_3, \end{aligned} \quad (\text{G.6})$$

where  $\vec{e}_1$ ,  $\vec{e}_2$  and  $\vec{e}_3$  are unit vectors in the  $w_1$ ,  $w_2$  and  $w_3$  directions, and the scale factors of the coordinate system are



$$h_i = \sqrt{\left(\frac{\partial x}{\partial w_i}\right)^2 + \left(\frac{\partial y}{\partial w_i}\right)^2 + \left(\frac{\partial z}{\partial w_i}\right)^2}, \quad i = 1,2,3. \quad (\text{G.7})$$

When we look for discontinuities in the gradient, divergence and rotor terms in Eqs. (A.1) and (B.1), the discontinuities originate from the derivatives of the field quantities and not from those of the  $h_i$  factors. Thus, in the discontinuity terms, all the  $h_i$  factors can be put outside the derivatives in Eq. (G.6) and all the derivatives are of the form  $\partial/(h_i \partial w_i)$ . If we think that the curvilinear coordinate system can be changed locally to a Cartesian one at every point on the surface, the derivatives  $\partial/(h_i \partial w_i)$  correspond to the local derivatives  $\partial/\partial x$ ,  $\partial/\partial y$  and  $\partial/\partial z$  point by point in this case. Eqs. (G.3) and (G.4) can therefore also be used with non-planar surface sources if  $\vec{e}_x$  is replaced by  $\vec{e}_1$ . Thus, the non-planar surface monopole distribution is

$$q_s = \left( \Delta \vec{U} + \frac{\Delta \rho}{\rho} \vec{U} \right) \cdot \vec{e}_1 \quad (\text{G.8})$$

and the non-planar surface dipole distribution is

$$\vec{F}_s = \Delta P \vec{e}_1 + \rho \vec{U} \cdot \vec{e}_1 \Delta \vec{U} - \vec{e}_1 \cdot \Delta \sigma_{\mu\mu}. \quad (\text{G.9})$$

## General equivalent Huygens sources

According to Huygens' principle, the effect of a radiating, absorbing or scattering (reflecting and/or diffracting) region  $V$  on the other part of the space can be taken into account by the equivalent Huygens sources distributed on any closed surface enclosing  $V$ , see, e.g., [6]. The equivalent source distributions have no effect inside the closed surface. Thus, the equivalent surface source distributions at surface  $S$  enclosing region  $V$ , see Figure G.1, are similar to those presented above, the discontinuities of the fields being replaced by the actual perturbation field values on the surface. The partition of the field quantities in static and perturbation components is presented in Appendix E, Eqs. (E.1), (E.2) and (E.3). Thus, the equivalent Huygens source distributions at the surface are obtained

from the fields at the surface, analogously with Eqs. (G.8) and (G.9), from equations

$$q_s = \left( \vec{u} + \frac{\rho'}{\rho} \vec{U} \right) \cdot \vec{e}_n \quad (\text{G.10})$$

$$\vec{f}_s = p \vec{e}_n + \rho \vec{U} \cdot \vec{e}_n \vec{u} - \vec{e}_n \cdot \overline{\overline{\sigma'}}_\mu, \quad (\text{G.11})$$

where  $\vec{e}_n$  is a unit normal vector outwards from the surface, see Figure G.1.

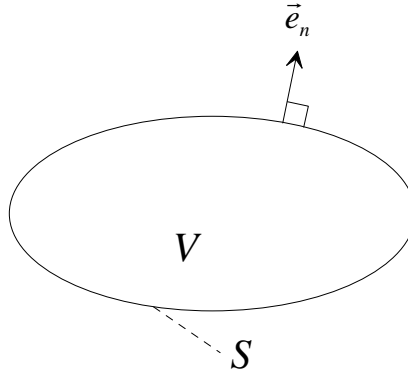


Figure G.1. Scattering object with volume  $V$  and surface  $S$ .

Supposing that the perturbation fields are much smaller than the static ones (the velocity does not have to obey this), only the first order terms of the Huygens source distributions can be used. Supposing that the thermal losses are small (this yielding to  $\rho' = p/c_0^2$ ,  $c_0$  is the first order value of  $c$ ) yields to the first order expressions from Eqs. (G.10) and (G.11)

$$q_s = \left( \vec{u} + \frac{1}{\rho_0 c_0^2} p \vec{U}_0 \right) \cdot \vec{e}_n \quad (\text{G.12})$$

$$\vec{f}_s = p \vec{e}_n + \rho_0 \vec{U}_0 \cdot \vec{e}_n \vec{u} - \vec{e}_n \cdot \overline{\overline{\sigma'}}_\mu. \quad (\text{G.13})$$

If the normal component of the static flow velocity disappears, the first order Huygens source distributions are

$$q_s = \bar{u} \cdot \bar{e}_n \quad (\text{G.14})$$

$$\vec{f}_s = p \bar{e}_n - \bar{e}_n \cdot \bar{\sigma}'_\mu. \quad (\text{G.15})$$

Thus, the effects of a scattering or absorbing surface on any sound field can be obtained by integrating the fields of the Huygens monopole and dipole distributions at the surface. The distributions to first order can be obtained from the actual fields at the surface using Eqs. (G.12) and (G.13), and with no normal static flow using Eqs. (G.14) and (G.15). If the Huygens surface is a true rigid surface, the normal component of the velocity disappears on the surface. In this case, the Huygens monopole distribution  $q_s$  also vanishes.

## Moving obstacles as Huygens sources

Let us suppose that the volume  $V$  in Figure G.1 is a moving obstacle having surface velocity  $\bar{v}$  at  $S$ . We can proceed in a similar way as before by representing the field variables by the sum of the static and perturbation components and replacing the discontinuities of the fields by the perturbation field variables at the surface. In this case, the static field  $\bar{U}_0$  should be replaced by  $\bar{U}_0 - \bar{v}$ . Now, we obtain from Eqs. (G.10) and (G.11), by neglecting second order terms having  $\rho' \bar{u}$  or  $\bar{u} \cdot \bar{e}_n \bar{u}$ ,

$$\begin{aligned} q_s &= \left[ \bar{u} + \frac{\rho'}{\rho} (\bar{U}_0 - \bar{v}) \right] \cdot \bar{e}_n \\ &= \frac{1}{\rho} [\rho \bar{u} + \rho' (\bar{U}_0 - \bar{v})] \cdot \bar{e}_n \\ &= \frac{1}{\rho} [\rho_0 \bar{v} + \rho (\bar{u} - \bar{v}) + \rho' \bar{U}_0] \cdot \bar{e}_n \end{aligned} \quad (\text{G.16})$$

$$\vec{f}_s = p \bar{e}_n + \rho (\bar{U}_0 - \bar{v}) \cdot \bar{e}_n \bar{u} - \bar{e}_n \cdot \bar{\sigma}'_\mu. \quad (\text{G.17})$$

If the normal component of the static flow velocity disappears, the Huygens source distributions at moving surface  $S$  are

$$q_s = \frac{1}{\rho} [\rho_0 \vec{v} + \rho(\vec{u} - \vec{v})] \cdot \vec{e}_n \quad (\text{G.18})$$

$$\vec{f}_s = p \vec{e}_n - \rho \vec{v} \cdot \vec{e}_n \vec{u} - \vec{e}_n \cdot \overline{\overline{\sigma}}'_\mu. \quad (\text{G.19})$$

In most practical cases, the body surface  $S$  can be assumed to be impermeable, in which case

$$(\vec{u} - \vec{v}) \cdot \vec{e}_n = 0, \quad (\text{G.20})$$

and the linearized equivalent source distributions are

$$q_s = \vec{v} \cdot \vec{e}_n \quad (\text{G.21})$$

$$\vec{f}_s = p \vec{e}_n - \vec{e}_n \cdot \overline{\overline{\sigma}}'_\mu. \quad (\text{G.22})$$

## Appendix H: Derivation of Lighthill's equation

Taking the time derivative of the equation of continuity (E.10) with the static fields extracted and the divergence of the Navier–Stokes equation (E.12) with the static fields extracted, the equations can be combined to yield

$$\begin{aligned} & \frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = \\ & = \nabla \nabla : \left( \rho \bar{U} \bar{U} - \rho_0 \bar{U}_0 \bar{U}_0 + \frac{\rho c^2 \beta T}{c_p} s \bar{\mathbf{I}} - \bar{\bar{\sigma}}'_\mu \right) + \\ & + \frac{\partial(\rho q)}{\partial t} - \nabla \cdot (\rho q \bar{U} + \bar{f}) + \nabla \nabla : \bar{T}. \end{aligned} \quad (\text{H.1})$$

If the mass, force and momentum source densities are absent, we have

$$\begin{aligned} & \frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = \\ & \nabla \nabla : \left( \rho \bar{U} \bar{U} - \rho_0 \bar{U}_0 \bar{U}_0 + \frac{\rho c^2 \beta T}{c_p} s \bar{\mathbf{I}} - \bar{\bar{\sigma}}'_\mu \right). \end{aligned} \quad (\text{H.2})$$

where the heat source distribution  $\varepsilon$  is also missing in the entropy perturbation  $s$ ; see Eq. (E.9).

When comparing the right-hand side of Eq. (H.2) with the last term on the right-hand side of Eq. (H.1), corresponding to the effect of the quadrupole distribution  $T$ , it can be noted that the terms inside the brackets on the right-hand side of Eq. (H.2) can be interpreted as quadrupole distribution  $\bar{T}_L$ . If the spatial derivatives of the static particle velocity are also small, at most of the perturbation order, Eq. (H.2) can be presented in Lighthill's form as

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = \nabla \nabla : \bar{T}_L, \quad (\text{H.3})$$

where Lighthill's turbulence stress dyadic (tensor) is

$$\overline{\overline{T}}_L = \rho \overline{\overline{U}} \overline{\overline{U}} + \frac{\rho c^2 \beta T}{c_p} s \overline{\overline{I}} - \overline{\overline{\sigma}}'_\mu. \quad (\text{H.4})$$

The sound pressure can be obtained from the integral (F.9) when using linearized equations. The integral has two options: the first one has spatial derivatives inside the integral and the second one outside the integral. To make the numerical computation easier and to minimize singularities in the integrals, it is advantageous to select the second alternative [13]. When using the second alternative, Green's function has to be spatially reciprocal according to the first equation in (F.4). Thus, with the assumptions presented, the sound pressure can be written as

$$p(\vec{r}, t) = \nabla \nabla : \int_{t_0=-\infty}^t \int_V \overline{\overline{T}}_L(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0, \quad (\text{H.5})$$

where volume  $V$  is principally the whole space, rather the volume where Lighthill's stress dyadic is non-zero and practically the volume containing all the regions in which Lighthill's stress dyadic is significant.

Next, it is assumed that the free space Green's function (F.2) can be used. The free space Green's function is spatially and temporally reciprocal. Using it, we obtain for the sound pressure

$$\begin{aligned} p(\vec{r}, t) &= \nabla \nabla : \int_{t_0=-\infty}^t \int_V \overline{\overline{T}}_L(\vec{r}_0, t_0) \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi |\vec{r} - \vec{r}_0|} dV dt_0 \\ &= \nabla \nabla : \int_V \frac{\overline{\overline{T}}_L(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c)}{4\pi |\vec{r} - \vec{r}_0|} dV. \end{aligned} \quad (\text{H.6})$$

## Appendix I: Derivation of Powell's equation

Powell's equation is an approximate version of Lighthill's equation. Assume that the two last terms in Lighthill's dyadic (H.4) vanish, namely the entropy perturbation and the perturbation component of the viscous part of the stress dyadic, which means that, also inside the source region, there are no viscous or thermal losses. Using Eqs. (R.16), (B.6) and (A.1) (without mass sources), we can write, in that case,

$$\begin{aligned}
 \overline{\nabla \nabla : T_L} &= \nabla \nabla : (\rho \vec{U} \vec{U}) = \nabla \cdot \left[ \nabla \cdot (\rho \vec{U} \vec{U}) \right] \\
 &= \nabla \cdot \left\{ (\nabla \rho) \cdot \vec{U} \vec{U} + \rho \left[ (\nabla \cdot \vec{U}) \vec{U} + \vec{U} \cdot (\nabla \vec{U}) \right] \right\} \\
 &= \nabla \cdot \left\{ \left[ \nabla \cdot (\rho \vec{U}) \right] \vec{U} + \rho \left[ \frac{1}{2} \nabla (\vec{U} \cdot \vec{U}) + \vec{\omega} \times \vec{U} \right] \right\} \\
 &= \nabla \cdot \left[ \rho \vec{\omega} \times \vec{U} + \frac{1}{2} \rho \nabla (\vec{U} \cdot \vec{U}) - \frac{\partial \rho}{\partial t} \vec{U} \right] \\
 &= \nabla \cdot \left[ \rho \vec{\omega} \times \vec{U} + \frac{1}{2} \nabla (\rho \vec{U} \cdot \vec{U}) - \frac{\partial \rho}{\partial t} \vec{U} - \frac{1}{2} (\nabla \rho) \vec{U} \cdot \vec{U} \right],
 \end{aligned} \tag{I.1}$$

where  $\vec{\omega}$  is the vorticity distribution according to Eq. (B.8). If it is further assumed that the fluid is incompressible (density variations with respect to time and spatial coordinates vanish) inside the source region, we obtain for Powell's equation

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = -\nabla \cdot \vec{f}_p, \tag{I.2}$$

where the source function is

$$\vec{f}_p = -\left[ \rho \vec{\omega} \times \vec{U} + \frac{1}{2} \nabla (\rho \vec{U} \cdot \vec{U}) \right]. \tag{I.3}$$

If the source function in Eq. (I.2) is compared with the second to last term on the right-hand side of Eq. (H.1), it can be interpreted as a dipole distribution.

The sound pressure can be obtained from integral (F.8) when using linearized equations. There are two options for the integral: the first one has spatial derivatives inside the integral and the second one outside the integral. To make the

numerical computation easier and to minimize singularities in the integrals, it is advantageous to select the second alternative [13]. When using the second alternative, Green's function has to be spatially reciprocal according to the first equation in (F.4). Thus, with the assumptions presented, the sound pressure can be written as

$$p(\vec{r}, t) = -\nabla \cdot \int_{t_0=-\infty}^t \int_V \vec{f}_p(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0. \quad (\text{I.4})$$

It is assumed that the free space Green's function (F.2) can be used. The free space Green's function is spatially and temporally reciprocal. By using it we obtain for the sound pressure

$$p(\vec{r}, t) = -\nabla \cdot \int_V \frac{\vec{f}_p(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dV. \quad (\text{I.5})$$



## Appendix J: Derivation of the Ffowcs Williams–Hawkings equation

The Ffowcs Williams–Hawkings equation is based on the same equations as Lighthill’s equation, the difference being that it takes into account the effects of moving boundaries by equivalent Huygens sources consisting of the surface monopole source distribution  $q_{ws}$  and the surface dipole source distribution  $\vec{f}_{ws}$ . Consider a body with volume  $V_c$  with an outer surface  $S$  moving in space and let the rest of the space volume, with  $V_c$  excluded, be denoted by  $V$ ; see Figure J.1.

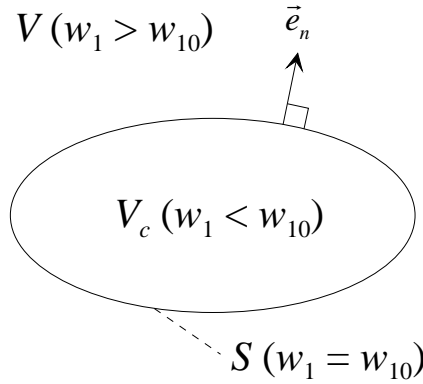


Figure J.1. Moving volume  $V_c$  with surface  $S$ , outer volume  $V$ .

The equivalent Huygens source distributions in the case of no normal component of the static flow at surface  $S$  can be obtained in linearized form from Eqs. (G.18) and (G.19) with Eq. (G.5). When these surface distributions are inserted into Eq. (H.1) and then proceeded similarly as was done to obtain Eq. (H.3), we can attain (omitting higher than second order quantities) an inhomogeneous wave equation valid for  $\rho'$  in  $V$

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = \frac{\partial [q_{ws} \delta(w_1 - w_{10})]}{\partial t} - \nabla \cdot [\vec{f}_{ws} \delta(w_1 - w_{10})] + \nabla \nabla : \left[ \overline{\overline{T}}_L H(w_1 - w_{10}) \right], \quad (J.1)$$

where

$$q_{WS} = [\rho_0 \vec{v} + \rho(\vec{u} - \vec{v})] \cdot \vec{e}_n \quad (\text{J.2})$$

$$\vec{f}_{WS} = p\vec{e}_n + \rho\vec{u}(\vec{u} - \vec{v}) \cdot \vec{e}_n - \vec{e}_n \cdot \overline{\overline{\sigma'}}_{\mu} \quad (\text{J.3})$$

$$\overline{\overline{T}}_L = \rho\vec{U}\vec{U} + \frac{\rho c^2 \beta T}{c_p} s\mathbf{I} - \overline{\overline{\sigma'}}_{\mu}, \quad (\text{J.4})$$

where  $\delta$  is the Dirac delta function,  $H$  is the Heaviside function according to Eq. (14),  $\vec{e}_n$  is a unit normal vector at  $S$  pointing outwards of  $V_c$ ,  $\vec{v}$  is the velocity of surface  $S$ , and the equivalent surface distributions are on surface  $S$  with  $w_1 = w_{10}$  in a coordinate system  $(w_1, w_2, w_3)$ , such that  $V_c$  is the region where  $w_1 < w_{10}$ . The Heaviside function with Lighthill's turbulence stress dyadic ensures that only that part of it existing in  $V$  is considered.

Next, the body surface  $S$  is considered to be impermeable, in which case we have, at the surface,

$$(\vec{u} - \vec{v}) \cdot \vec{e}_n = 0, \quad (\text{J.5})$$

and Eqs. (J.2) and (J.3) reduce to

$$q_{WS} = \rho_0 \vec{v} \cdot \vec{e}_n \quad (\text{J.6})$$

$$\vec{f}_{WS} = p\vec{e}_n - \vec{e}_n \cdot \overline{\overline{\sigma'}}_{\mu}. \quad (\text{J.7})$$

The term  $q_{WS}$  represents the equivalent Huygens monopole distribution at surface  $S$ . If the motion  $\vec{v}$  is such that the total volume  $V_c$  remains constant, which is equivalent to the total volume velocity  $Q$  out of surface  $S$  being zero, it is not reasonable to handle its surface as a monopole distribution. Let us further define that  $\vec{v}$  represents the velocity not only at the surface  $S$  but also inside  $S$  for the whole  $V_c$ . The time rate of change of volume of an element is proportional to the divergence of the velocity; see, e.g., [20]. This does not lead to the conclusion, argued by Goldstein [5], that the divergence disappears point by point in a proper coordinate system. The total volume velocity  $Q$  through  $S$  can be obtained by integrating the normal component of the velocity at  $S$ , which can fur-

ther be written using the Gauss theorem as a volume integral of the divergence of the velocity inside  $V_c$  as

$$Q = \oint_S \vec{v} \cdot \vec{e}_n \, dS = \int_{V_c} \nabla \cdot \vec{v} \, dV = 0. \quad (\text{J.8})$$

Thus, the demand for the total volume  $V_c$  to remain constant leads to the conclusion that only the mean value of the divergence of the velocity disappears inside  $V_c$ . By looking at the basic version of the equation of continuity (A.1) without the source terms, it can be said for fluids that if the deformation inside  $V_c$  is incompressible (density does not vary with time), in the Lagrangian coordinate system  $\vec{\zeta}$  (moving with individual particles), we have in  $V_c$

$$\nabla_{\vec{\zeta}} \cdot \vec{v} = 0. \quad (\text{J.9})$$

The same conclusion can also be obtained for solid materials, see, e.g., [21].

If the divergence of the velocity disappears, the surface monopole distribution  $q_{ws}$  can be replaced by volume dipole distribution  $\vec{f}_{wvc}$  and volume quadrupole distribution  $T_{wvc}$  inside  $V_c$ , according to Appendix P [12, 5], where

$$\vec{f}_{wvc} = \rho_0 \frac{d\vec{v}}{dt} = \rho_0 \vec{a} \quad (\text{J.10})$$

$$\overline{\overline{T}}_{wvc} = \rho_0 \vec{v} \vec{v}, \quad (\text{J.11})$$

where  $\vec{a}$  is the acceleration in the Lagrangian coordinate system (acceleration of the individual particles inside  $V_c$ ). In this case, Eq. (J.1) for density perturbation fields in  $V$  can now be written as

$$\begin{aligned} \frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = & -\nabla \cdot \left\{ \vec{f}_{wvc} [1 - H(w_1 - w_{10})] \right\} \\ & + \nabla \nabla : \left\{ \overline{\overline{T}}_{wvc} [1 - H(w_1 - w_{10})] \right\} \\ - \nabla \cdot \left[ \vec{f}_{ws} \delta(w_1 - w_{10}) \right] & + \nabla \nabla : \left[ \overline{\overline{T}}_L H(w_1 - w_{10}) \right], \end{aligned} \quad (\text{J.12})$$

where the dipole volume distribution  $\underline{\underline{f}}_{wVc}$  in  $V_c$  can be obtained from Eq. (J.10), the quadrupole volume distribution  $T_{wVc}$  in  $V_c$  can be obtained from Eq. (J.11), the dipole surface distribution  $\underline{\underline{f}}_{wS}$  at surface  $S$  can be obtained from Eq. (J.7) and the quadrupole volume distribution  $T_L$  in  $V$  can be obtained from Eq. (J.4).

The total sound pressure in  $V$  can be obtained from the sum of the effects of various source parts in Eq. (J.12) as

$$P = P_{fVc} + P_{TVc} + P_{fS} + P_L, \quad (\text{J.13})$$

where the sound pressure components can be obtained from integrals (F.8), (F.9), (F.12) and (F.9) when using linearized equations. There are two options for integrals: the first has spatial derivatives inside the integral and the second one outside the integral. To make the numerical computation easier and to minimize singularities in the integrals, it is advantageous to select the second options [13]. When using the second alternatives, Green's function has to be spatially reciprocal according to the first equation in (F.4). So with the assumptions presented, the various sound pressure components in Eq. (J.13) can be written as

$$P_{fVc}(\vec{r}, t) = -\nabla \cdot \int_{t_0=-\infty}^t \int_{V_c(t_0)} \underline{\underline{f}}_{wVc}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \quad (\text{J.14})$$

$$P_{TVc}(\vec{r}, t) = \nabla \nabla : \int_{t_0=-\infty}^t \int_{V_c(t_0)} \overline{\overline{T}}_{wVc}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 \quad (\text{J.15})$$

$$P_{fS}(\vec{r}, t) = -\nabla \cdot \int_{t_0=-\infty}^t \int_{S(t_0)} \underline{\underline{f}}_{wS}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0 \quad (\text{J.16})$$

$$P_L(\vec{r}, t) = \nabla \nabla : \int_{t_0=-\infty}^t \int_{V(t_0)} \overline{\overline{T}}_L(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0. \quad (\text{J.17})$$

If the surface monopole distribution  $q_{wS}$  in Eq. (J.6) is not replaced by volume dipole and quadrupole distributions, the total sound pressure in  $V$  can be obtained from the sum of the effects of various source parts in Eq. (J.1) as

$$p = p_{vS} + p_{fS} + p_L, \quad (\text{J.18})$$

where  $p_{fS}$  and  $p_L$  are as before and can be obtained from Eqs. (J.16) and (J.17). The first term  $p_{vS}$  in Eq. (J.18) can be obtained using Eq. (F.11). The integral has three options: the first two have time derivatives inside the integral and the third one outside the integral. To make the numerical computation easier and to minimize singularities in the integrals, it is advantageous to select the third option [13]. When using the third alternative, Green's function has to be reciprocal temporally according to the second equation in (F.4). Thus, the first term in Eq. (J.18) can be written as

$$p_{vS}(\vec{r}, t) = \frac{\partial}{\partial t} \int_{t_0=-\infty}^t \int_{S(t_0)} \rho_0 q_{WS}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0. \quad (\text{J.19})$$

Next, it is assumed that the free space Green's function (F.2) can be used. This assumes that all boundaries in the space should be treated similarly to  $S$ . The free space Green's function is spatially and temporally reciprocal. Using it, we obtain, for the various partial sound fields

$$p_{vS}(\vec{r}, t) = \frac{\partial}{\partial t} \int_{t_0=-\infty}^{\infty} \int_{S(t_0)} \rho_0 q_{WS}(\vec{r}_0, t_0) \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dS dt_0 \quad (\text{J.20})$$

$$p_{fVc}(\vec{r}, t) = -\nabla \cdot \int_{t_0=-\infty}^{\infty} \int_{V_c(t_0)} \vec{f}_{wVc}(\vec{r}_0, t_0) \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dV dt_0 \quad (\text{J.21})$$

$$p_{TVc}(\vec{r}, t) = \nabla \nabla : \int_{t_0=-\infty}^{\infty} \int_{V_c(t_0)} \overline{\overline{T}}_{wVc}(\vec{r}_0, t_0) \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dV dt_0 \quad (\text{J.22})$$

$$p_{fS}(\vec{r}, t) = -\nabla \cdot \int_{t_0=-\infty}^{\infty} \int_{S(t_0)} \vec{f}_{WS}(\vec{r}_0, t_0) \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dS dt_0 \quad (\text{J.23})$$

$$p_L(\vec{r}, t) = \nabla \nabla : \int_{t_0=-\infty}^{\infty} \int_{V(t_0)} \overline{\overline{T}}_L(\vec{r}_0, t_0) \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi|\vec{r} - \vec{r}_0|} dV dt_0, \quad (\text{J.24})$$

where the upper time limits of the time integrations have been extended to infinite. This is allowable because the causality of Green's function causes Green's function to vanish if  $t_0 > t$ .

Next, it is supposed that the body with volume  $V_c$  is a solid particle retaining its shape, and its movement consists of translation and rotation. It is beneficial to transform the Cartesian coordinate system with vectors  $\vec{r}_0$  to a moving Cartesian coordinate system with vectors  $\vec{\zeta}(\vec{r}_0, t_0)$  such that, in this new coordinate system, the body will not move, i.e., to Lagrangian coordinates. The terms  $\vec{v}$  and  $\vec{a}$  in this coordinate systems signify the convectonal velocity and acceleration, i.e., the velocity and acceleration of individual points when the coordinate movement is temporally fictitiously stopped

$$\begin{aligned}\vec{v}(\vec{\zeta}, t_0) &= \left. \frac{\partial \vec{r}_0(\vec{\zeta}, t_0)}{\partial t_0} \right|_{\vec{\zeta}=\text{constant}} \\ \vec{a}(\vec{\zeta}, t_0) &= \left. \frac{\partial \vec{v}(\vec{\zeta}, t_0)}{\partial t_0} \right|_{\vec{\zeta}=\text{constant}}.\end{aligned}\tag{J.25}$$

This new coordinate system will translate and rotate in a similar way to the body  $V_c$  itself. Then because in this coordinate system the limits of the volume and surface integrals are independent of time  $t_0$ , the order of integrations can be changed, integrating first with respect to time. Using identity [5]

$$\int_{-\infty}^{\infty} f(t_0) \delta[h(t_0)] dt_0 = \sum_i \frac{f(t_{e,i})}{\left| \frac{dh(t_0)}{dt_0} \right|_{t_0=t_{e,i}}},\tag{J.26}$$

where  $t_{e,i}$  is the  $i$ th real root of

$$h(t_0) = 0,\tag{J.27}$$

we can see that in the integrals we are developing ( $c$  substituted by its linearized value  $c_0$ )

$$\begin{aligned}
 h(t_0) &= t_0 - t + |\vec{r} - \vec{r}_0| / c_0 \\
 t_e &= t - |\vec{r} - \vec{r}_0| / c_0 \\
 \frac{dh}{dt_0} &= 1 - \frac{\vec{r} - \vec{r}_0}{c_0 |\vec{r} - \vec{r}_0|} \cdot \frac{\partial \vec{r}_0}{\partial t_0},
 \end{aligned} \tag{J.28}$$

and, using also the first equation of (J.25), that the time integrations in the changed spatial coordinates produce to the denominators of the expressions the Doppler factor

$$C(\vec{r}, \vec{\zeta}, t_e) = \left| 1 - \frac{\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)}{|\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)|} \cdot \frac{\vec{v}(\vec{\zeta}, t_e)}{c_0} \right|. \tag{J.29}$$

Thus, after the time integration, we obtain for the partial sound fields [12, 5]

$$p_{vs}(\vec{r}, t) = \frac{\partial}{\partial t} \int_S \frac{\rho_0 q_{ws}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dS(\vec{\zeta}) \tag{J.30}$$

$$p_{fvc}(\vec{r}, t) = -\nabla \cdot \int_{V_c} \frac{\vec{f}_{wvc}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dV(\vec{\zeta}) \tag{J.31}$$

$$p_{Tvc}(\vec{r}, t) = \nabla \nabla : \int_{V_c} \frac{\overline{\overline{T}}_{wvc}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dV(\vec{\zeta}) \tag{J.32}$$

$$p_{fs}(\vec{r}, t) = -\nabla \cdot \int_S \frac{\vec{f}_{ws}(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dS(\vec{\zeta}) \tag{J.33}$$

$$p_L(\vec{r}, t) = \nabla \nabla : \int_V \frac{\overline{\overline{T}}_L(\vec{\zeta}, t_e)}{4\pi |\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)| C(\vec{r}, \vec{\zeta}, t_e)} dV(\vec{\zeta}), \tag{J.34}$$

where the retarded time  $t_e = t_e(\vec{r}, t, \vec{\zeta})$  is the solution to the equation

$$h(t_e, t, \vec{r}, \vec{\zeta}) = t_e - t + \frac{|\vec{r} - \vec{r}_0(\vec{\zeta}, t_e)|}{c_0} = 0. \tag{J.35}$$

If there is more than one solution to Eq. (J.35), a sum of all such solutions has to be used [13, 5]. This is the case at supersonic speeds. At subsonic speeds,  $S^*$ ,  $V_c^*$  and  $V^*$  are the physical surface and volumes. At supersonic speeds, they are functions of  $\vec{r}$  and  $t$  because the whole physical regions do not contribute to the integrals [13] (Eq. (J.35) has no solutions for some values of  $\bar{\zeta}$ ).

The coordinate transformation normally requires the Jacobians of the transformation to be taken into account for surface and volume integrals. As the absolute value of the Jacobian in the transformations between Cartesian coordinates is unity, the elementary areas  $dS$  and elementary volumes  $dV$  remain the same in these kinds of transformations, so no Jacobians are needed in the integrals [13, 5].

At speeds at which the quantity  $C$  approaches one ( $v/c_0 \approx 0$ ), the solution fails due to singularities in the integrals. Other types of coordinate transformations are available to overcome this problem [12, 13].



## Appendix K: Derivation of Curle's equation

If the surface  $S$  in the Ffowcs Williams–Hawkings equation is assumed to be rigid and it does not move, the equivalent monopole surface source distribution  $q_{WS}$  and its substitute volume dipole and quadrupole source distributions  $\vec{f}_{wvc}$  and  $T_{wvc}$  disappear as well as the corresponding density perturbation components  $\rho'_{vS}$ ,  $\rho'_{fvc}$  and  $\rho'_{Tvc}$ . This leads to Curle's equation

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = -\nabla \cdot [\vec{f}_{WS} \delta(w_1 - w_{10})] + \nabla \nabla : [\overline{\overline{T}}_L H(w_1 - w_{10})], \quad (\text{K.1})$$

where

$$\vec{f}_{WS} = p \vec{e}_n - \vec{e}_n \cdot \overline{\overline{\sigma}}'_{\mu} \quad (\text{K.2})$$

$$\overline{\overline{T}}_L = \rho \overline{\overline{U}} \overline{\overline{U}} + \frac{\rho c^2 \beta T}{c_p} s \overline{\overline{I}} - \overline{\overline{\sigma}}'_{\mu}. \quad (\text{K.3})$$

The equivalent dipole surface source distribution takes care of the sound scattering caused by the stationary surface.

The total sound pressure in  $V$  can be obtained from the sum of the effects of various source parts in Eq. (K.1) as

$$p = p_{fS} + p_L, \quad (\text{K.4})$$

where according to Eqs. (J.16) and (J.17)

$$p_{fS}(\vec{r}, t) = -\nabla \cdot \int_{t_0=-\infty}^t \int_{S(t_0)} \vec{f}_{WS}(\vec{r}_0, t_0) g(\vec{r}, t | \vec{r}_0, t_0) dS dt_0 \quad (\text{K.5})$$

$$p_L(\vec{r}, t) = \nabla \nabla : \int_{t_0=-\infty}^{\infty} \int_{V(t_0)} \overline{\overline{T}}_L(\vec{r}_0, t_0) \frac{\delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)}{4\pi |\vec{r} - \vec{r}_0|} dV dt_0. \quad (\text{K.6})$$

It is assumed that the free space Green's function (F.2) can be used. Using this, we obtain for the various partial sound fields from Eqs. (J.23) and (J.24)

$$p_{fs}(\vec{r}, t) = -\nabla \cdot \int_S \frac{\vec{f}_{ws}(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c_0)}{4\pi|\vec{r} - \vec{r}_0|} dS \quad (\text{K.7})$$

$$p_L(\vec{r}, t) = \nabla \nabla : \int_V \frac{\overline{T}_L(\vec{r}_0, t - |\vec{r} - \vec{r}_0|/c_0)}{4\pi|\vec{r} - \vec{r}_0|} dV . \quad (\text{K.8})$$

## Appendix L: Derivation of Phillips' equation

Taking the Lagrangian time derivative of the equation of continuity (A.2) of an ideal gas and the divergence of the Navier–Stokes equation (B.9) of an ideal gas and using vector identity (Q.1), we obtain for field quantity  $\Pi$

$$\begin{aligned} \frac{d^2 \Pi}{dt^2} - \nabla \cdot (c^2 \nabla \Pi) &= (\nabla \vec{U})_T : (\nabla \vec{U}) + \frac{d}{dt} \left( \frac{1}{c_p} \frac{dS}{dT} \right) \\ &\quad - \nabla \cdot \left[ \frac{1}{\rho} \left( \nabla \cdot \overset{=}{\sigma}_\mu + \vec{F} - \nabla \cdot \overset{=}{T} \right) \right] + \frac{dq}{dt}, \end{aligned} \quad (\text{L.1})$$

where subscript ‘T’ denotes the transpose of a dyadic, see Eq. (R.11), and the field quantity  $\Pi$ , scaled logarithmic pressure, is defined in Eq. (C.12).

In the case of no mass, force and momentum distributions, this leads to the Phillips' equation

$$\begin{aligned} &\frac{d^2 \Pi}{dt^2} - \nabla \cdot (c^2 \nabla \Pi) \\ &= (\nabla \vec{U})_T : (\nabla \vec{U}) + \frac{d}{dt} \left( \frac{1}{c_p} \frac{dS}{dT} \right) - \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \overset{=}{\sigma}_\mu \right). \end{aligned} \quad (\text{L.2})$$



## Appendix M: Derivation of Lilley's equation

If Eq. (L.1) is derivated in the Lagrangian way, we obtain further

$$\begin{aligned} \frac{d}{dt} \left[ \frac{d^2 \Pi}{dt^2} - \nabla \cdot (c^2 \nabla \Pi) \right] &= 2(\nabla \vec{U})_T : \frac{d}{dt} (\nabla \vec{U}) + \frac{d^2}{dt^2} \left( \frac{1}{c_p} \frac{dS}{dT} \right) \\ &\quad - \frac{d}{dt} \left\{ \nabla \cdot \left[ \frac{1}{\rho} \left( \nabla \cdot \bar{\bar{\sigma}}_\mu + \vec{F} - \nabla \cdot \bar{\bar{T}} \right) \right] \right\} + \frac{d^2 q}{dt^2}. \end{aligned} \quad (\text{M.1})$$

Using identity (Q.2) and the divergence of the Navier–Stokes equation version (B.9) for ideal gases, we have

$$\begin{aligned} \frac{d}{dt} (\nabla \vec{U}) &= \nabla \frac{d\vec{U}}{dt} - (\nabla \vec{U}) \cdot (\nabla \vec{U}) \\ &= \nabla \left[ -c^2 \nabla \Pi + \frac{1}{\rho} \left( \nabla \cdot \bar{\bar{\sigma}}_\mu + \vec{F} - \nabla \cdot \bar{\bar{T}} \right) \right] - (\nabla \vec{U}) \cdot (\nabla \vec{U}). \end{aligned} \quad (\text{M.2})$$

Now, Eq. (M.1) can be written as

$$\begin{aligned} \frac{d}{dt} \left[ \frac{d^2 \Pi}{dt^2} - \nabla \cdot (c^2 \nabla \Pi) \right] &+ 2(\nabla \vec{U})_T : \nabla (c^2 \nabla \Pi) = \\ &= -2(\nabla \vec{U})_T : [(\nabla \vec{U}) \cdot (\nabla \vec{U})] + \Psi + \\ &+ \frac{d^2 q}{dt^2} - \frac{d}{dt} \left\{ \nabla \cdot \left[ \frac{1}{\rho} \left( \vec{F} - \nabla \cdot \bar{\bar{T}} \right) \right] \right\} + 2(\nabla \vec{U})_T : \nabla \left[ \frac{1}{\rho} \left( \vec{F} - \nabla \cdot \bar{\bar{T}} \right) \right], \end{aligned} \quad (\text{M.3})$$

where

$$\Psi = 2(\nabla \vec{U})_T : \nabla \left( \frac{1}{\rho} \nabla \cdot \bar{\bar{\sigma}}_\mu \right) - \frac{d}{dt} \left[ \nabla \cdot \left( \frac{1}{\rho} \nabla \cdot \bar{\bar{\sigma}}_\mu \right) \right] + \frac{d^2}{dt^2} \left( \frac{1}{c_p} \frac{dS}{dT} \right). \quad (\text{M.4})$$

In the case of no mass, force and momentum source distributions, Eq. (M.3) leads to Lilley's equation

$$\begin{aligned} \frac{d}{dt} \left[ \frac{d^2 \Pi}{dt^2} - \nabla \cdot (c^2 \nabla \Pi) \right] + 2(\nabla \vec{U})_r : \nabla (c^2 \nabla \Pi) \\ = -2(\nabla \vec{U})_r : [(\nabla \vec{U}) \cdot (\nabla \vec{U})] + \Psi. \end{aligned} \quad (\text{M.5})$$

## Appendix N: Derivation of Howe's equation

Taking the divergence of the Navier–Stokes equation of form (B.11) and the partial time derivative of the continuity equation version (A.5), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{\partial}{\partial t} \left( \frac{\beta T}{c_p} \frac{dS}{dt} \right) \\ + \frac{\partial q}{\partial t} - \nabla \cdot \left[ \frac{1}{\rho} \left( \nabla \cdot \vec{\sigma}_\mu + \vec{F} - \nabla \cdot \vec{T} \right) \right]. \end{aligned} \quad (\text{N.1})$$

By adding the same parts to both sides of this equation, it can be formally presented as

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\ + \left\{ \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \frac{\partial}{\partial t} \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) \right\} \\ + \frac{\partial}{\partial t} \left( \frac{\beta T}{c_p} \frac{dS}{dt} \right) + \frac{\partial q}{\partial t} - \nabla \cdot \left[ \frac{1}{\rho} \left( \nabla \cdot \vec{\sigma}_\mu + \vec{F} - \nabla \cdot \vec{T} \right) \right]. \end{aligned} \quad (\text{N.2})$$

From Eq. (B.11), we can obtain for the middle term in brackets { } in Eq. (N.2) above

$$\begin{aligned} \frac{d\vec{U}}{dt} \cdot \nabla B = - \frac{\partial \vec{U}}{\partial t} \cdot \frac{d\vec{U}}{dt} - \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\ + \frac{1}{\rho} \frac{d\vec{U}}{dt} \cdot \left( \nabla \cdot \vec{\sigma}_\mu + \vec{F} - \nabla \cdot \vec{T} \right). \end{aligned} \quad (\text{N.3})$$

Combining the first and third terms in brackets { } in Eq. (N.2), we obtain, using Eq. (A.8) (lower one),

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) - \frac{\partial}{\partial t} \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) = \\
 & = \frac{d}{dt} \left[ \frac{1}{c^2} \left( \frac{1}{\rho} \frac{dP}{dt} + T \frac{dS}{dt} + \vec{U} \cdot \frac{d\vec{U}}{dt} \right) \right] - \frac{\partial}{\partial t} \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) \\
 & = \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} + \frac{1}{c^2} \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) + \\
 & \quad + \vec{U} \cdot \nabla \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) - \frac{\partial}{\partial t} \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) = \\
 & = \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} + \frac{1}{c^2} \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \vec{U} \cdot \nabla \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right).
 \end{aligned} \tag{N.4}$$

Now, Eq. (N.2) can be presented, with the help of Eqs. (N.3) and (N.4), as

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\
 & - \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{\beta T}{c_p} \frac{dS}{dt} \right) \\
 & + \frac{d}{dt} \left( \frac{1}{c^2} \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \vec{U} \cdot \nabla \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) - \frac{1}{c^2} \frac{\partial \vec{U}}{\partial t} \cdot \frac{d\vec{U}}{dt} \\
 & + \frac{\partial q}{\partial t} + \frac{1}{\rho c^2} \frac{d\vec{U}}{dt} \cdot \left( \nabla \cdot \vec{\sigma}_\mu + \vec{F} - \nabla \cdot \vec{T} \right) - \nabla \cdot \left[ \frac{1}{\rho} \left( \nabla \cdot \vec{\sigma}_\mu + \vec{F} - \nabla \cdot \vec{T} \right) \right].
 \end{aligned} \tag{N.5}$$

The third line in (N.5) can be further developed as below, using identity (Q.3),



$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \vec{U} \cdot \nabla \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) - \frac{1}{c^2} \frac{\partial \vec{U}}{\partial t} \cdot \frac{d\vec{U}}{dt} \\
 = & \frac{d}{dt} \left( \frac{1}{c^2} \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \vec{U} \cdot \frac{d}{dt} \nabla \left( \frac{1}{\rho c^2} P \right) + \vec{U} \cdot \nabla \vec{U} \cdot \nabla \left( \frac{1}{\rho c^2} P \right) - \\
 & - \vec{U} \cdot \nabla \left[ \frac{d}{dt} \left( \frac{1}{\rho c^2} P \right) \right] - \frac{1}{c^2} \frac{\partial \vec{U}}{\partial t} \cdot \frac{d\vec{U}}{dt} = \\
 = & \frac{d}{dt} \left\{ \vec{U} \cdot \left[ \frac{1}{c^2} \frac{d\vec{U}}{dt} + \nabla \left( \frac{1}{\rho c^2} P \right) \right] \right\} - \frac{d\vec{U}}{dt} \cdot \left[ \frac{1}{c^2} \frac{d\vec{U}}{dt} + \nabla \left( \frac{1}{\rho c^2} P \right) \right] + \\
 + & \frac{d\vec{U}}{dt} \cdot \left( \frac{1}{c^2} \vec{U} \cdot \nabla \vec{U} \right) + \vec{U} \cdot \nabla \vec{U} \cdot \nabla \left( \frac{1}{\rho c^2} P \right) - \vec{U} \cdot \nabla \left[ \frac{d}{dt} \left( \frac{1}{\rho c^2} P \right) \right] = \\
 = & \vec{U} \cdot \frac{d}{dt} \left[ \frac{1}{c^2} \frac{d\vec{U}}{dt} + \nabla \left( \frac{1}{\rho c^2} P \right) \right] + \vec{U} \cdot \nabla \vec{U} \cdot \left[ \frac{1}{c^2} \frac{d\vec{U}}{dt} + \nabla \left( \frac{1}{\rho c^2} P \right) \right] - \\
 & - \vec{U} \cdot \nabla \left[ \frac{d}{dt} \left( \frac{1}{\rho c^2} P \right) \right] = \\
 = & \left( \vec{U} \cdot \frac{d}{dt} + \vec{U} \cdot \nabla \vec{U} \cdot \right) \left[ \frac{1}{c^2} \frac{d\vec{U}}{dt} + \nabla \left( \frac{1}{\rho c^2} P \right) \right] - \vec{U} \cdot \nabla \left[ \frac{d}{dt} \left( \frac{1}{\rho c^2} P \right) \right]. \tag{N.6}
 \end{aligned}$$

With the help of Eq. (B.1), we can write

$$\begin{aligned}
 \frac{1}{c^2} \frac{d\vec{U}}{dt} + \nabla \left( \frac{1}{\rho c^2} P \right) &= \frac{1}{\rho c^2} \left( \rho \frac{d\vec{U}}{dt} + \nabla P \right) + \nabla \left( \frac{1}{\rho c^2} P \right) \\
 &= \frac{1}{\rho c^2} \left( \nabla \cdot \vec{\sigma}_\mu + \vec{F} - \nabla \cdot \vec{T} \right) + \nabla \left( \frac{1}{\rho c^2} P \right). \tag{N.7}
 \end{aligned}$$

Thus, Eq. (N.6) can be further developed into

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \vec{U} \cdot \frac{d\vec{U}}{dt} \right) + \vec{U} \cdot \nabla \left( \frac{1}{\rho c^2} \frac{dP}{dt} \right) - \frac{1}{c^2} \frac{\partial \vec{U}}{\partial t} \cdot \frac{d\vec{U}}{dt} \\
 &= \left( \vec{U} \cdot \frac{d}{dt} + \vec{U} \cdot \nabla \vec{U} \cdot \right) \left[ \frac{1}{\rho c^2} \left( \nabla \cdot \overline{\overline{\sigma}}_\mu + \vec{F} - \nabla \cdot \overline{\overline{T}} \right) \right] \\
 &+ \left( \vec{U} \cdot \frac{d}{dt} + \vec{U} \cdot \nabla \vec{U} \cdot \right) \nabla \left( \frac{1}{\rho c^2} \right) P - \vec{U} \cdot \nabla \left[ \frac{d}{dt} \left( \frac{1}{\rho c^2} \right) P \right].
 \end{aligned} \tag{N.8}$$

Eq. (N.5) can now be written, with the help of Eq. (N.8), as

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\
 & - \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{\beta T}{c_p} \frac{dS}{dt} \right) \\
 & + \frac{\partial q}{\partial t} + \frac{1}{\rho c^2} \frac{d\vec{U}}{dt} \cdot \left( \nabla \cdot \overline{\overline{\sigma}}_\mu + \vec{F} - \nabla \cdot \overline{\overline{T}} \right) \\
 & - \nabla \cdot \left[ \frac{1}{\rho} \left( \nabla \cdot \overline{\overline{\sigma}}_\mu + \vec{F} - \nabla \cdot \overline{\overline{T}} \right) \right] \\
 & + \left( \vec{U} \cdot \frac{d}{dt} + \vec{U} \cdot \nabla \vec{U} \cdot \right) \left[ \frac{1}{\rho c^2} \left( \nabla \cdot \overline{\overline{\sigma}}_\mu + \vec{F} - \nabla \cdot \overline{\overline{T}} \right) \right] \\
 & + \left( \vec{U} \cdot \frac{d}{dt} + \vec{U} \cdot \nabla \vec{U} \cdot \right) \nabla \left( \frac{1}{\rho c^2} \right) P - \vec{U} \cdot \nabla \left[ \frac{d}{dt} \left( \frac{1}{\rho c^2} \right) P \right].
 \end{aligned} \tag{N.9}$$

If the fluid is assumed to have constant compressibility  $1/\rho c^2$  with respect to time and spatial coordinates, we obtain

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\
 & - \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{\beta T}{c_p} \frac{dS}{dt} \right) + \frac{\partial q}{\partial t} \\
 & + \left\{ \frac{1}{\rho c^2} \left[ \vec{U} \cdot \frac{d}{dt} + \left( \frac{d\vec{U}}{dt} + \vec{U} \cdot \nabla \vec{U} \cdot \right) \right] - \nabla \cdot \frac{1}{\rho} \right\} \left( \nabla \cdot \overline{\overline{\sigma}}_\mu + \vec{F} - \nabla \cdot \overline{\overline{T}} \right).
 \end{aligned} \tag{N.10}$$

Without any mass, force and momentum source distributions, this leads to

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\
 & - \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{\beta T}{c_p} \frac{dS}{dt} \right) \\
 & + \left\{ \frac{1}{\rho c^2} \left[ \vec{U} \cdot \frac{d}{dt} + \left( \frac{d\vec{U}}{dt} + \vec{U} \cdot \nabla \vec{U} \right) \cdot \right] - \nabla \cdot \frac{1}{\rho} \right\} \left( \nabla \cdot \vec{\sigma}_\mu \right).
 \end{aligned} \tag{N.11}$$

Furthermore, without viscous forces, this leads to

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\
 & - \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{\beta T}{c_p} \frac{dS}{dt} \right).
 \end{aligned} \tag{N.12}$$

With an ideal gas, Eq. (C.9) holds for the coefficient of thermal expansion, and Eq. (N.12) can be written in Howe's form

$$\begin{aligned}
 & \frac{d}{dt} \left( \frac{1}{c^2} \frac{dB}{dt} \right) + \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot \nabla B - \nabla^2 B = \nabla \cdot (\vec{\omega} \times \vec{U} - T \nabla S) \\
 & - \frac{1}{c^2} \frac{d\vec{U}}{dt} \cdot (\vec{\omega} \times \vec{U} - T \nabla S) + \frac{d}{dt} \left( \frac{T}{c^2} \frac{dS}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{1}{c_p} \frac{dS}{dt} \right).
 \end{aligned} \tag{N.13}$$



## Appendix O: Derivation of Doak's equation

Taking the time derivative of the continuity equation version (A.12) for an ideal gas and the divergence of the Navier–Stokes equation version (B.13), we obtain, also using Eq. (A.7)

$$\begin{aligned} \nabla^2 B = & -\frac{\partial q}{\partial t} - \nabla \cdot (\bar{\omega} \times \bar{U} - \bar{V}) - \frac{\partial}{\partial t} \left( \frac{1}{R} \frac{dS}{dt} \right) \\ & + \frac{\partial}{\partial t} \left( \frac{1}{c^2} \right) \frac{dH}{dt} + \frac{1}{c^2} \frac{\partial}{\partial t} \left( \frac{dB}{dt} - \bar{U} \cdot \frac{d\bar{U}}{dt} \right). \end{aligned} \quad (\text{O.1})$$

The last term on the right-hand side of the second line in Eq. (O.1) will be dealt with next.

Using Eq. (B.13), we can write

$$\begin{aligned} \frac{dB}{dt} - \bar{U} \cdot \frac{d\bar{U}}{dt} &= \frac{\partial B}{\partial t} + \bar{U} \cdot \nabla B - \bar{U} \cdot \left( \frac{\partial \bar{U}}{\partial t} + \bar{U} \cdot \nabla \bar{U} \right) \\ &= \frac{\partial B}{\partial t} + \bar{U} \cdot \nabla B + \bar{U} \cdot (\nabla B + \bar{\omega} \times \bar{U} - \bar{V} - \bar{U} \cdot \nabla \bar{U}) \\ &= \frac{\partial B}{\partial t} + 2\bar{U} \cdot \nabla B - \bar{U} \cdot \bar{V} - \bar{U} \bar{U} : \nabla \bar{U}, \end{aligned} \quad (\text{O.2})$$

where we have also used that

$$\bar{U} \cdot (\bar{\omega} \times \bar{U}) = 0. \quad (\text{O.3})$$

By further using Eq. (B.13), the time derivative of Eq. (O.2) can be written as

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left( \frac{d\mathbf{B}}{dt} - \vec{U} \cdot \frac{d\vec{U}}{dt} \right) = \\
 & = \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{B}}{\partial t} + 2\vec{U} \cdot \nabla \mathbf{B} - \vec{U} \cdot \vec{V} - \vec{U}\vec{U} : \nabla \vec{U} \right) = \\
 & = \frac{\partial^2 \mathbf{B}}{\partial t^2} + 2 \frac{\partial \vec{U}}{\partial t} \cdot \nabla \mathbf{B} + 2\vec{U} \cdot \frac{\partial}{\partial t} (\nabla \mathbf{B}) \\
 & - \frac{\partial}{\partial t} (\vec{U}\vec{U}) : \nabla \vec{U} - \vec{U}\vec{U} : \frac{\partial}{\partial t} (\nabla \vec{U}) - \frac{\partial \vec{U}}{\partial t} \cdot \vec{V} - \vec{U} \cdot \frac{\partial \vec{V}}{\partial t} = \\
 & = \frac{\partial^2 \mathbf{B}}{\partial t^2} + 2\vec{U} \cdot \frac{\partial}{\partial t} (\nabla \mathbf{B}) + 2 \frac{\partial \vec{U}}{\partial t} \cdot \nabla \mathbf{B} - \left( \frac{\partial \vec{U}}{\partial t} \vec{U} + \vec{U} \frac{\partial \vec{U}}{\partial t} \right) : \nabla \vec{U} + \\
 & + \vec{U}\vec{U} : \nabla (\nabla \mathbf{B} + \vec{\omega} \times \vec{U} - \vec{V}) - \frac{\partial \vec{U}}{\partial t} \cdot \vec{V} - \vec{U} \cdot \frac{\partial \vec{V}}{\partial t} = \\
 & = \frac{\partial^2 \mathbf{B}}{\partial t^2} + 2\vec{U} \cdot \frac{\partial}{\partial t} (\nabla \mathbf{B}) + \vec{U}\vec{U} : \nabla \nabla \mathbf{B} - \vec{U} \cdot \frac{\partial \vec{V}}{\partial t} + \vec{U}\vec{U} : \nabla (\vec{\omega} \times \vec{U} - \vec{V}) + \\
 & + \frac{\partial \vec{U}}{\partial t} \cdot \left\{ 2\nabla \mathbf{B} - \vec{U} \cdot \left[ \nabla \vec{U} + (\nabla \vec{U})_{\text{r}} \right] \right\} - \frac{\partial \vec{U}}{\partial t} \cdot \vec{V}. \tag{O.4}
 \end{aligned}$$

Using dyadic identities (R.16) (fourth equation) and (R.8) (fourth equation), we notice that

$$\vec{a} \cdot (\nabla \vec{a})_{\text{r}} = \vec{a} \cdot \nabla \vec{a} + \vec{a} \cdot (\nabla \times \vec{a}) \times \vec{I} = \vec{a} \cdot \nabla \vec{a} - (\nabla \times \vec{a}) \times \vec{a}. \tag{O.5}$$

Using this and Eqs. (B.6) and (B.8), we can write

$$\vec{U} \cdot \left[ \nabla \vec{U} + (\nabla \vec{U})_{\text{r}} \right] = 2\vec{U} \cdot \nabla \vec{U} - (\nabla \times \vec{U}) \times \vec{U} = \nabla (\vec{U} \cdot \vec{U}) + \vec{\omega} \times \vec{U}. \tag{O.6}$$

Using Eq. (O.6) above and the definition of the stagnation enthalpy (A.7), the first term in the last line of Eq. (O.4) can be presented as

$$\begin{aligned}
 & \frac{\partial \vec{U}}{\partial t} \cdot \left\{ 2\nabla \mathbf{B} - \vec{U} \cdot \left[ \nabla \vec{U} + (\nabla \vec{U})_{\text{r}} \right] \right\} = \frac{\partial \vec{U}}{\partial t} \cdot \left[ 2\nabla \mathbf{B} - \nabla (\vec{U} \cdot \vec{U}) - \vec{\omega} \times \vec{U} \right] \\
 & = \frac{\partial \vec{U}}{\partial t} \cdot \left[ 2(\nabla \mathbf{B} - \frac{1}{2} \vec{U} \cdot \vec{U}) - \vec{\omega} \times \vec{U} \right] = \frac{\partial \vec{U}}{\partial t} \cdot \left[ 2\nabla H - \vec{\omega} \times \vec{U} \right]. \tag{O.7}
 \end{aligned}$$

Using Eqs. (B.13) and (O.7), the last line of Eq. (O.4) can be written as

$$\begin{aligned}
 & \frac{\partial \vec{U}}{\partial t} \cdot \left\{ 2\nabla B - \vec{U} \cdot \left[ \nabla \vec{U} + (\nabla \vec{U})_r \right] \right\} - \frac{\partial \vec{U}}{\partial t} \cdot \vec{V} \\
 & = (-\nabla B - \vec{\omega} \times \vec{U} + \vec{V}) \cdot (2\nabla H - \vec{\omega} \times \vec{U} - \vec{V}) \\
 & = \nabla B \cdot (\vec{\omega} \times \vec{U} + \vec{V} - 2\nabla H) + (\vec{\omega} \times \vec{U} + \vec{V} - 2\nabla H) \cdot (\vec{\omega} \times \vec{U} - \vec{V}).
 \end{aligned} \tag{O.8}$$

Now, Eq. (O.4) is

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left( \frac{dB}{dt} - \vec{U} \cdot \frac{d\vec{U}}{dt} \right) \\
 & = \frac{\partial^2 B}{\partial t^2} + \left( 2\vec{U} \frac{\partial}{\partial t} + \vec{\omega} \times \vec{U} + \vec{V} - 2\nabla H \right) \cdot \nabla B + \vec{U}\vec{U} : \nabla \nabla B \\
 & \quad + \left[ \vec{U}\vec{U} : \nabla + (\vec{\omega} \times \vec{U} + \vec{V} - 2\nabla H) \cdot \right] (\vec{\omega} \times \vec{U} - \vec{V}) - \vec{U} \cdot \frac{\partial \vec{V}}{\partial t}.
 \end{aligned} \tag{O.9}$$

Finally, inserting Eqs. (O.2) and (O.9) into Eq. (O.1), we obtain, by rearranging the terms,

$$\begin{aligned}
 & \nabla^2 B - \frac{1}{c^2} \left[ \frac{\partial^2}{\partial t^2} + \vec{U}\vec{U} : \nabla \nabla + \left( 2\vec{U} \frac{\partial}{\partial t} + \vec{\omega} \times \vec{U} + \vec{V} - 2\nabla H \right) \cdot \nabla \right] B \\
 & = \left\{ -(\nabla \cdot) + \frac{1}{c^2} \left[ \vec{U}\vec{U} : \nabla + (\vec{\omega} \times \vec{U} + \vec{V} - 2\nabla H) \cdot \right] \right\} (\vec{\omega} \times \vec{U} - \vec{V}) \\
 & \quad + \frac{\partial}{\partial t} \left( \frac{1}{c^2} \right) \frac{dH}{dt} - \frac{\partial}{\partial t} \left( \frac{1}{R} \frac{dS}{dt} \right) - \frac{1}{c^2} \vec{U} \cdot \frac{\partial \vec{V}}{\partial t} - \frac{\partial q}{\partial t}.
 \end{aligned} \tag{O.10}$$

Expressing the stagnation enthalpy as the sum of its temporal mean value (line over) and the fluctuating part ( $\cdot$ )

$$B = \bar{B} + B' \tag{O.11}$$

and seeing from Eq. (B.13) that

$$\nabla \bar{B} = \overline{(-\bar{\omega} \times \bar{U} + \bar{V})} - \overline{\left(\frac{\partial \bar{U}'}{\partial t}\right)}, \quad (\text{O.12})$$

we obtain for the fluctuating part of the stagnation enthalpy from Eq. (O.10)

$$\begin{aligned} \nabla^2 B' - \left\{ \frac{1}{c^2} \left[ \frac{\partial^2}{\partial t^2} + \bar{U}\bar{U} : \nabla \nabla + \left( 2\bar{U} \frac{\partial}{\partial t} + \bar{\omega} \times \bar{U} + \bar{V} - 2\nabla H \right) \cdot \nabla \right] B' \right\}' \\ = \left( \left\{ -(\nabla \cdot) + \frac{1}{c^2} \left[ \bar{U}\bar{U} : \nabla + (\bar{\omega} \times \bar{U} + \bar{V} - 2\nabla H) \cdot \right] \right\} \right)' \\ \left[ \left( \bar{\omega} \times \bar{U} \right)' - \bar{V}' - \overline{\left(\frac{\partial \bar{U}'}{\partial t}\right)} \right]' \\ + \left[ \frac{\partial}{\partial t} \left( \frac{1}{c^2} \right)' \frac{dH}{dt} \right]' - \left[ \frac{\partial}{\partial t} \left( \frac{1}{R} \frac{dS}{dt} \right)' \right]' - \left[ \frac{1}{c^2} \bar{U} \cdot \frac{\partial \bar{V}'}{\partial t} \right]' - \frac{\partial q}{\partial t}, \end{aligned} \quad (\text{O.13})$$

where superscript  $'$  means the fluctuating part and line over means temporal mean value.

If the mass and momentum source distribution vanish, Eq. (O.10) can be written in Doak's form as

$$\begin{aligned} \nabla^2 B' - \left\{ \frac{1}{c^2} \left[ \frac{\partial^2}{\partial t^2} + \bar{U}\bar{U} : \nabla \nabla + \left( 2\bar{U} \frac{\partial}{\partial t} + \bar{\omega} \times \bar{U} + \bar{V} - 2\nabla H \right) \cdot \nabla \right] B' \right\}' \\ = \left( \left\{ -(\nabla \cdot) + \frac{1}{c^2} \left[ \bar{U}\bar{U} : \nabla + (\bar{\omega} \times \bar{U} + \bar{V} - 2\nabla H) \cdot \right] \right\} \right)' \\ \left[ \left( \bar{\omega} \times \bar{U} \right)' - \bar{V}' - \overline{\left(\frac{\partial \bar{U}'}{\partial t}\right)} \right]' \\ + \left[ \frac{\partial}{\partial t} \left( \frac{1}{c^2} \right)' \frac{dH}{dt} \right]' - \left[ \frac{\partial}{\partial t} \left( \frac{1}{R} \frac{dS}{dt} \right)' \right]' - \left[ \frac{1}{c^2} \bar{U} \cdot \frac{\partial \bar{V}'}{\partial t} \right]', \end{aligned} \quad (\text{O.14})$$



where, now,

$$\vec{V} = T\nabla S + \frac{1}{\rho} \left( \nabla \cdot \vec{\sigma}_\mu + \vec{F} \right). \quad (\text{O.15})$$



# Appendix P: Reduction of the volume displacement term to dipole and quadrupole terms

## Derivation of the basic equation

Suppose that function  $f$  obeys

$$\nabla_0 f(R, t_0) = -\nabla f(R, t_0), \quad (\text{P.1})$$

where  $t_0$  is the 'source time' variable,  $R$  is

$$R = |\vec{r} - \vec{r}_0|, \quad (\text{P.2})$$

where  $\vec{r}$  is a field point vector and  $\vec{r}_0$  is a source point vector. Gradient  $\nabla$  operates on the field coordinates and gradient  $\nabla_0$  operates on the source coordinates.

Suppose that the particle velocity is only a function of source coordinates

$$\vec{U} = \vec{U}(\vec{r}_0). \quad (\text{P.3})$$

This leads to relations

$$\begin{aligned} \nabla \cdot \vec{U} &= 0 \\ \nabla \vec{U} &= 0 \\ \nabla \times \vec{U} &= 0. \end{aligned} \quad (\text{P.4})$$

Suppose further that the particle velocity obeys

$$\nabla_0 \cdot \vec{U} = 0 \quad (\text{P.5})$$

Based on these assumptions, using expression (Q.4) and identities (Q.5), we can write

Appendix P: Reduction of the volume displacement term to dipole and quadrupole terms

$$\begin{aligned}
& \nabla_0 \cdot \left[ \vec{U} \left( \frac{\partial f}{\partial t_0} + \vec{U} \cdot \nabla f \right) \right] + \frac{\partial}{\partial t_0} (\vec{U} \cdot \nabla f) \\
&= \vec{U} \cdot \nabla_0 \frac{\partial f}{\partial t_0} + \vec{U} \cdot \nabla_0 (\vec{U} \cdot \nabla f) + \frac{\partial}{\partial t_0} (\vec{U} \cdot \nabla f) \\
&= -\vec{U} \cdot \nabla \frac{\partial f}{\partial t_0} + \frac{d}{dt_0} (\vec{U} \cdot \nabla f) \\
&= -\vec{U} \cdot \nabla \frac{\partial f}{\partial t_0} + \frac{d\vec{U}}{dt_0} \cdot \nabla f + \vec{U} \cdot \frac{d}{dt_0} (\nabla f) \\
&= -\vec{U} \cdot \nabla \frac{\partial f}{\partial t_0} + \nabla \cdot \left( f \frac{d\vec{U}}{dt_0} \right) + \vec{U} \cdot \nabla \frac{df}{dt_0} \\
&= -\vec{U} \cdot \nabla \frac{\partial f}{\partial t_0} + \nabla \cdot \left( f \frac{d\vec{U}}{dt_0} \right) + \vec{U} \cdot \nabla \left( \frac{\partial f}{\partial t_0} + \vec{U} \cdot \nabla_0 f \right) \\
&= \nabla \cdot \left( f \frac{d\vec{U}}{dt_0} \right) - \vec{U} \cdot \nabla (\vec{U} \cdot \nabla f) \\
&= \nabla \cdot \left( f \frac{d\vec{U}}{dt_0} \right) - \vec{U} \cdot [\vec{U} \cdot \nabla \nabla f + \vec{U} \times \nabla \times \nabla f + ] \\
&= \nabla \cdot \left( f \frac{d\vec{U}}{dt_0} \right) - \vec{U} \vec{U} : \nabla \nabla f .
\end{aligned} \tag{P.6}$$

Based on the same assumptions as above and using Eq. (R.16) (seventh line), we can also write

$$\nabla \nabla : (\vec{U} \vec{U} f) = \nabla \cdot \nabla \cdot (\vec{U} \vec{U} f) = \nabla \cdot \nabla f \cdot \vec{U} \vec{U} = \vec{U} \vec{U} : \nabla \nabla f . \tag{P.7}$$

Comparing Eqs. (P.6) and (P.7), we notice that

$$\begin{aligned}
& \nabla_0 \cdot \left[ \vec{U} \left( \frac{\partial f}{\partial t_0} + \vec{U} \cdot \nabla f \right) \right] + \frac{\partial}{\partial t_0} (\vec{U} \cdot \nabla f) \\
&= \nabla \cdot \left( f \frac{d\vec{U}}{dt_0} \right) - \nabla \nabla : (\vec{U} \vec{U} f) .
\end{aligned} \tag{P.8}$$

Let  $S(t_0)$  be a closed surface enclosing volume  $V(t_0)$  and  $\vec{e}_n$  be a unit vector pointing outwards from the volume at the surface; see Figure P.1.

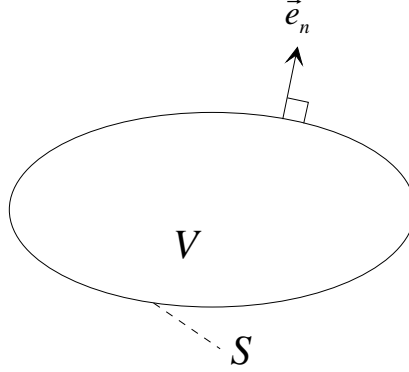


Figure P.1. Volume  $V$  surrounded by surface  $S$ .

Suppose that  $f(R, t_0)$  vanishes outside time interval  $-T < t_0 < T$ . In this case,

$$\begin{aligned}
 0 &= \left[ \int_{V(t_0)} (\vec{U} \cdot \nabla f) dV \right]_{t_0=T} - \left[ \int_{V(t_0)} (\vec{U} \cdot \nabla f) dV \right]_{t_0=-T} \\
 &= \int_{-T}^T \frac{d}{dt_0} \int_{V(t_0)} (\vec{U} \cdot \nabla f) dV dt_0.
 \end{aligned} \tag{P.9}$$

Then, applying Leibniz's rule [5, 19]

$$\begin{aligned}
 &\frac{d}{dt_0} \int_{V(t_0)} (\vec{U} \cdot \nabla f) dV \\
 &= \int_{V(t_0)} \frac{\partial}{\partial t_0} (\vec{U} \cdot \nabla f) dV + \oint_{S(t_0)} (\vec{U} \cdot \nabla f) (\vec{U} \cdot \vec{e}_n) dS
 \end{aligned} \tag{P.10}$$

and the Gauss theorem, we obtain

Appendix P: Reduction of the volume displacement term to dipole and quadrupole terms

$$\begin{aligned}
0 &= \int_{-T}^T \frac{d}{dt_0} \int_{V(t_0)} (\vec{U} \cdot \nabla f) dV dt_0 \\
&= \int_{-T}^T \int_{V(t_0)} \frac{\partial}{\partial t_0} (\vec{U} \cdot \nabla f) dV dt_0 + \int_{-T}^T \oint_{S(t_0)} (\vec{U} \cdot \nabla f) (\vec{U} \cdot \vec{e}_n) dS dt_0 \\
&= \int_{-T}^T \int_{V(t_0)} \left\{ \frac{\partial}{\partial t_0} (\vec{U} \cdot \nabla f) + \nabla_0 \cdot [(\vec{U} \cdot \nabla f) \vec{U}] \right\} dV dt_0.
\end{aligned} \tag{P.11}$$

Using the Gauss' theorem, we can also write

$$\int_{-T}^T \oint_{S(t_0)} \vec{U} \cdot \vec{e}_n \frac{\partial f}{\partial t_0} dS dt_0 = \int_{-T}^T \int_{V(t_0)} \nabla_0 \cdot \left( \vec{U} \frac{\partial f}{\partial t_0} \right) dV dt_0. \tag{P.12}$$

By adding zero to the equation above, with the help of Eq. (P.11), we obtain

$$\begin{aligned}
&\int_{-T}^T \oint_{S(t_0)} \vec{U} \cdot \vec{e}_n \frac{\partial f}{\partial t_0} dS dt_0 \\
&= \int_{-T}^T \int_{V(t_0)} \left\{ \nabla_0 \cdot \left[ \vec{U} \left( \frac{\partial f}{\partial t_0} + \vec{U} \cdot \nabla f \right) \right] + \frac{\partial}{\partial t_0} (\vec{U} \cdot \nabla f) \right\} dV dt_0.
\end{aligned} \tag{P.13}$$

Taking Eq. (P.8) into account, the equation can be written as

$$\begin{aligned}
&\int_{-T}^T \oint_{S(t_0)} \vec{U} \cdot \vec{e}_n \frac{\partial f}{\partial t_0} dS dt_0 \\
&= \nabla \cdot \int_{-T}^T \int_{V(t_0)} f \frac{d\vec{U}}{dt_0} dV dt_0 - \nabla \nabla : \int_{-T}^T \int_{V(t_0)} \vec{U} \vec{U} f dV dt_0.
\end{aligned} \tag{P.14}$$

## Application to the Huygens monopole sources

Let  $S(t_0)$  be a physical impermeable surface enclosing volume  $V(t_0)$  and let the particle velocity be  $\vec{v}$  inside  $V$  and at  $S$ . If function  $f$  is related to Green's function  $g$  by

$$f(R, t_0) = \rho_0 g(\vec{r}, t | \vec{r}_0, t_0), \quad (\text{P.15})$$

we can write, using Eq. (P.14),

$$\begin{aligned} & - \int_{t_0=-\infty}^t \oint_{S(t_0)} \rho_0 \vec{v} \cdot \vec{e}_n \frac{\partial g(\vec{r}, t | \vec{r}_0, t_0)}{\partial t_0} dS dt_0 = \\ & = -\nabla \cdot \int_{t_0=-\infty}^t \int_{V(t_0)} \rho_0 \frac{d\vec{v}}{dt_0} g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0 + \\ & + \nabla \nabla : \int_{t_0=-\infty}^t \int_{V(t_0)} \rho_0 \vec{v} \vec{v} g(\vec{r}, t | \vec{r}_0, t_0) dV dt_0. \end{aligned} \quad (\text{P.16})$$

The upper time integration limit  $T$  in Eq. (P.14) has been changed to time  $t$ , and the lower limit has been changed to  $-\infty$ . The former is reasonable because the causality of Green's function demands that Green's function vanishes if  $t_0 > t$ . The latter is reasonable because function  $f$  is assumed to vanish below  $-T$ .

Comparing the left-hand side of Eq. (P.16) with Eq. (F.11) (middle version) and Eq. (G.21), it can be noted that it represents the sound radiation of the equivalent Huygens monopole source distribution at moving surface  $S$ . Comparing the first term on the right-hand side of Eq. (P.16) with Eq. (F.8) (second version), we notice that it represents the sound radiation of a dipole volume distribution located inside  $V$ . Furthermore, comparing the second term on the right-hand side of Eq. (P.16) with Eq. (F.9) (second version), it can be noted that it represents the sound radiation of a quadrupole volume distribution located inside  $V$ . All the sound radiations occur outwards from volume  $V$ . Let surface  $S(t_0)$  be represented by  $w_1 = w_{10}(t_0)$  in a curvilinear coordinate system  $(w_1, w_2, w_3)$  such that  $V$  is the region in which  $w_1 \leq w_{10}(t_0)$ . In conclusion, with the assumptions used, the equivalent Huygens' monopole source distribution

$$\begin{aligned} q &= q_S \delta(w_1 - w_{10}(t_0)) \\ q_S &= \rho_0 \vec{v} \cdot \vec{e}_n, \end{aligned} \quad (\text{P.17})$$

radiating outwards from volume  $V$  and located at an impermeable physical surface  $S$  can be replaced by dipole and quadrupole volume source distributions

$$\begin{aligned}\vec{f} &= \rho_0 \frac{d\vec{v}}{dt_0} [1 - H(w_1 - w_{10}(t_0))] \\ \vec{T} &= \rho_0 \vec{v} \vec{v} [1 - H(w_1 - w_{10}(t_0))],\end{aligned}\tag{P.18}$$

radiating outwards from volume  $V$  and located inside  $V$ .  $H$  is the Heaviside function according to Eq. (14).

The first assumption (P.1) to obtain this result is equivalent to the spatial reciprocity of Green's function, according to Eqs. (P.15) and (F.4). The second assumption (P.5) demands the divergence of the particle velocity to disappear inside  $V$ . The third assumption demands that Green's function is non-zero only in some limited time interval. The use of the concept of the equivalent Huygens monopole sources in the present way demands that the fields can be divided into static and linearized perturbation fields and that the normal component of the static flow velocity is zero at surface  $S$ .



## Appendix Q: Some useful identities

Three useful identities used in this report:

$$\nabla \cdot \frac{d\vec{a}}{dt} = \frac{d}{dt}(\nabla \cdot \vec{a}) + (\nabla \vec{U})_T : (\nabla \vec{a}) \quad (\text{Q.1})$$

$$\nabla \frac{d\vec{a}}{dt} = \frac{d}{dt}(\nabla \vec{a}) + (\nabla \vec{U}) \cdot (\nabla \vec{a}) \quad (\text{Q.2})$$

$$\nabla \frac{da}{dt} = \frac{d}{dt}(\nabla a) + (\nabla \vec{U}) \cdot (\nabla a), \quad (\text{Q.3})$$

where subscript ‘T’ denotes the transpose of a dyadic; see Eq. (R.11).

If  $\vec{r}$  is a field point vector,  $\vec{r}_0$  is a source point vector,  $t$  is the time related to the field points,  $t_0$  is the time related to the source points, gradient  $\nabla$  operates on the field coordinates and gradient  $\nabla_0$  operates on the source coordinates, the last identity can also be presented by the formula

$$\nabla \frac{da}{dt_0} = \frac{d}{dt_0}(\nabla a) + (\nabla \vec{U}) \cdot (\nabla_0 a). \quad (\text{Q.4})$$

The identities are derived next. In the derivations, vector identities

$$\begin{aligned} \nabla(\vec{a} \cdot \vec{b}) &= \vec{a} \cdot \nabla \vec{b} + \vec{b} \cdot \nabla \vec{a} + \vec{a} \times \nabla \times \vec{b} + \vec{b} \times \nabla \times \vec{a} \\ \nabla \times \nabla \phi &= 0 \end{aligned} \quad (\text{Q.5})$$

and their dyadic equivalences, see Eq. (R.16),

$$\begin{aligned} \nabla(\vec{a} \cdot \vec{A}) &= \vec{a} \cdot \nabla \vec{A} + \vec{A}_T \cdot \nabla \vec{a} + \vec{a} \times \nabla \times \vec{A} + \vec{A}_T \times \nabla \times \vec{a} \\ \nabla \times \nabla \vec{a} &= 0 \end{aligned} \quad (\text{Q.6})$$

are needed, as well as the vector identity

$$\vec{a} \times \nabla \times \vec{b} = -(\nabla \times \vec{b}) \times \vec{a} = [\nabla \vec{b} - (\nabla \vec{b})_{\text{T}}] \cdot \vec{a} \quad (\text{Q.7})$$

and its dyadic equivalence

$$\overline{\overline{A}} \times \nabla \times \vec{a} = \overline{\overline{A}} \cdot [(\nabla \vec{a})_{\text{T}} - \nabla \vec{a}]. \quad (\text{Q.8})$$

Dyadic equivalence (Q.8) can be obtained by substituting  $\overline{\overline{A}} = \vec{b} \vec{c}$  and using the vector identity (Q.7).

The following dyadic relationships are also needed, see, Eq. (R.16),

$$\begin{aligned} \nabla \cdot (\vec{a} \cdot \overline{\overline{A}}) &= \nabla \cdot (\overline{\overline{A}}_{\text{T}} \cdot \vec{a}) = \vec{a} \cdot (\nabla \cdot \overline{\overline{A}}_{\text{T}}) + \overline{\overline{A}}_{\text{T}} : \nabla \vec{a} \\ \nabla \cdot (\nabla \vec{a})_{\text{T}} &= \nabla (\nabla \cdot \vec{a}). \end{aligned} \quad (\text{Q.9})$$

## Derivation of identity (Q.1)

By noting that

$$\begin{aligned} \nabla \cdot \frac{d\vec{a}}{dt} &= \nabla \cdot \left( \frac{\partial \vec{a}}{\partial t} + \vec{U} \cdot \nabla \vec{a} \right) = \nabla \cdot \frac{\partial \vec{a}}{\partial t} + \nabla \cdot (\vec{U} \cdot \nabla \vec{a}) \\ \frac{d}{dt} (\nabla \cdot \vec{a}) &= \frac{\partial}{\partial t} (\nabla \cdot \vec{a}) + \vec{U} \cdot \nabla (\nabla \cdot \vec{a}), \end{aligned} \quad (\text{Q.10})$$

we can write

$$\nabla \cdot \frac{d\vec{a}}{dt} = \frac{d}{dt} (\nabla \cdot \vec{a}) + \nabla \cdot (\vec{U} \cdot \nabla \vec{a}) - \vec{U} \cdot \nabla \nabla \cdot \vec{a}. \quad (\text{Q.11})$$

By using identity (Q.9), we can see that

$$\begin{aligned} &\nabla \cdot (\vec{U} \cdot \nabla \vec{a}) - \vec{U} \cdot \nabla \nabla \cdot \vec{a} \\ &= \vec{U} \cdot \nabla \cdot (\nabla \vec{a})_{\text{T}} + (\nabla \vec{a})_{\text{T}} : \nabla \vec{U} - \vec{U} \cdot \nabla \cdot (\nabla \vec{a})_{\text{T}} \\ &= (\nabla \vec{a})_{\text{T}} : \nabla \vec{U} = (\nabla \vec{U})_{\text{T}} : \nabla \vec{a}. \end{aligned} \quad (\text{Q.12})$$

By comparing Eqs. (Q.11) and (Q.12), identity (Q.1) can readily be written.

## Derivation of identity (Q.2)

By noting that

$$\begin{aligned}\nabla \frac{d\vec{a}}{dt} &= \nabla \left( \frac{\partial \vec{a}}{\partial t} + \vec{U} \cdot \nabla \vec{a} \right) = \nabla \frac{\partial \vec{a}}{\partial t} + \nabla (\vec{U} \cdot \nabla \vec{a}) \\ \frac{d}{dt} (\nabla \vec{a}) &= \frac{\partial}{\partial t} (\nabla \vec{a}) + \vec{U} \cdot \nabla (\nabla \vec{a}),\end{aligned}\tag{Q.13}$$

we can write

$$\begin{aligned}\nabla \frac{d\vec{a}}{dt} &= \frac{d}{dt} (\nabla \vec{a}) + \nabla (\vec{U} \cdot \nabla \vec{a}) - \vec{U} \cdot \nabla \nabla \vec{a} \\ &= \frac{d}{dt} (\nabla \vec{a}) + \vec{U} \cdot \nabla \nabla \vec{a} + (\nabla \vec{a})_{\text{T}} \cdot (\nabla \vec{U}) \\ &\quad + \vec{U} \times \nabla \times (\nabla \vec{a}) + (\nabla \vec{a})_{\text{T}} \times \nabla \times \vec{U} - \vec{U} \cdot \nabla \nabla \vec{a} \\ &= \frac{d}{dt} (\nabla \vec{a}) + (\nabla \vec{a})_{\text{T}} \cdot (\nabla \vec{U}) + (\nabla \vec{a})_{\text{T}} \times \nabla \times \vec{U},\end{aligned}\tag{Q.14}$$

where identities (Q.6) have been used. Using identity (Q.8), we note that

$$\begin{aligned}& (\nabla \vec{a})_{\text{T}} \cdot (\nabla \vec{U}) + (\nabla \vec{a})_{\text{T}} \times \nabla \times \vec{U} \\ &= (\nabla \vec{a})_{\text{T}} \cdot (\nabla \vec{U}) + (\nabla \vec{a})_{\text{T}} \cdot [(\nabla \vec{U})_{\text{T}} - \nabla \vec{U}] \\ &= (\nabla \vec{U})_{\text{T}} \cdot (\nabla \vec{a}).\end{aligned}\tag{Q.15}$$

By comparing Eqs. (Q.14) and (Q.15), identity (Q.2) can readily be written.

### Derivation of identity (Q.3)

By noting that

$$\begin{aligned}\nabla \frac{da}{dt} &= \nabla \left( \frac{\partial a}{\partial t} + \vec{U} \cdot \nabla a \right) = \nabla \frac{\partial a}{\partial t} + \nabla (\vec{U} \cdot \nabla a) \\ \frac{d}{dt} (\nabla a) &= \frac{\partial}{\partial t} (\nabla a) + \vec{U} \cdot \nabla (\nabla a),\end{aligned}\tag{Q.16}$$

we can write

$$\begin{aligned}\nabla \frac{da}{dt} &= \frac{d}{dt} (\nabla a) + \nabla (\vec{U} \cdot \nabla a) - \vec{U} \cdot \nabla \nabla a \\ &= \frac{d}{dt} (\nabla a) + \vec{U} \cdot \nabla \nabla a + (\nabla a) \cdot (\nabla \vec{U}) + \\ &\quad + \vec{U} \times \nabla \times (\nabla a) + (\nabla a) \times \nabla \times \vec{U} - \vec{U} \cdot \nabla \nabla a \\ &= \frac{d}{dt} (\nabla a) + (\nabla a) \cdot (\nabla \vec{U}) + (\nabla a) \times \nabla \times \vec{U},\end{aligned}\tag{Q.17}$$

where identities (Q.5) have been used. By using identity (Q.7), we can note that

$$\begin{aligned}(\nabla a) \cdot (\nabla \vec{U}) + (\nabla a) \times \nabla \times \vec{U} \\ = (\nabla a) \cdot (\nabla \vec{U}) + [\nabla \vec{U} - (\nabla \vec{U})_T] \cdot (\nabla a) \\ = (\nabla \vec{U}) \cdot (\nabla a).\end{aligned}\tag{Q.18}$$

By comparing Eqs. (Q.17) and (Q.18), identity (Q.3) can readily be written.

### Derivation of identity (Q.4)

By noting that

$$\begin{aligned}\nabla \frac{da}{dt_0} &= \nabla \left( \frac{\partial a}{\partial t_0} + \vec{U} \cdot \nabla_0 a \right) = \nabla \frac{\partial a}{\partial t_0} + \nabla (\vec{U} \cdot \nabla_0 a) \\ \frac{d}{dt_0} (\nabla a) &= \frac{\partial}{\partial t_0} (\nabla a) + \vec{U} \cdot \nabla_0 (\nabla a),\end{aligned}\tag{Q.19}$$

we can write

$$\begin{aligned}
\nabla \frac{da}{dt_0} &= \frac{d}{dt_0}(\nabla a) + \nabla(\vec{U} \cdot \nabla_0 a) - \vec{U} \cdot \nabla_0 \nabla a \\
&= \frac{d}{dt_0}(\nabla a) + \vec{U} \cdot \nabla \nabla_0 a + (\nabla_0 a) \cdot (\nabla \vec{U}) + \\
&\quad + \vec{U} \times \nabla \times (\nabla_0 a) + (\nabla_0 a) \times \nabla \times \vec{U} - \vec{U} \cdot \nabla_0 \nabla a \\
&= \frac{d}{dt_0}(\nabla a) + (\nabla_0 a) \cdot (\nabla \vec{U}) + (\nabla_0 a) \times \nabla \times \vec{U},
\end{aligned} \tag{Q.20}$$

where identities (Q.5) have been used. By using identity (Q.7), we can note that

$$\begin{aligned}
&(\nabla_0 a) \cdot (\nabla \vec{U}) + (\nabla_0 a) \times \nabla \times \vec{U} \\
&= (\nabla_0 a) \cdot (\nabla \vec{U}) + [\nabla \vec{U} - (\nabla \vec{U})_{\text{T}}] \cdot (\nabla_0 a) \\
&= (\nabla \vec{U}) \cdot (\nabla_0 a).
\end{aligned} \tag{Q.21}$$

By comparing Eqs. (Q.20) and (Q.21), identity (Q.4) can readily be written.



## Appendix R: Basics of dyadic notation

More comprehensive presentations of dyadic notation can be found in, e.g., Refs. [22], [23], [24] and [19].

Dyadic is one way to present a linear mapping in the vector space. The starting point of the dyadic is a bilinear function from two vector arguments called the dyadic product or dyad and marked by empty

$$\vec{a}, \vec{b} \rightarrow \overline{\vec{a}\vec{b}}. \quad (\text{R.1})$$

The dyadic is a polynom formed from dyadic products. Here, the dyadic is marked by two lines above as

$$\overline{\overline{A}} = \overline{\vec{a}\vec{b}} + \overline{\vec{c}\vec{d}} + \overline{\vec{e}\vec{f}}. \quad (\text{R.2})$$

An arbitrary dyadic can always be presented by the same number of dyadic products as the number of the dimensions of the vector space.

A linear mapping that can be presented with dyadics can always be presented with matrices or tensors. Let us have a linear mapping presented with a matrix as

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \quad (\text{R.3})$$

This mapping can be presented with the tensor presentation as

$$f_i = a_{ij} x_j. \quad (\text{R.4})$$

With dyadics this mapping is presented as

$$\vec{f} = \overline{\overline{a}} \cdot \vec{x}. \quad (\text{R.5})$$

The multiplying dyadic can be presented as, e.g.,

$$\vec{a} = \sum_{i=1}^3 \sum_{j=1}^3 a_{ij} \vec{e}_i \vec{e}_j, \quad (\text{R.6})$$

where  $\vec{e}_i$  and  $\vec{e}_j$  are the unit vectors in the directions of the  $i$ - and  $j$ -axes. The terms  $a_{ij}$  in Eq (R.6) are just the same as in Eqs. (R.3) and (R.4).

Normally, the dyadic product does not commute

$$\vec{a}\vec{b} \neq \vec{b}\vec{a}. \quad (\text{R.7})$$

The following are examples of dyadic mappings

$$\begin{aligned} \vec{b} &= (\vec{d}\vec{c}) \cdot \vec{a} = \vec{d}(\vec{c} \cdot \vec{a}) \\ \vec{b} &= \vec{a} \cdot (\vec{d}\vec{c}) = (\vec{a} \cdot \vec{d})\vec{c} \\ \vec{b} &= \vec{\bar{I}} \cdot \vec{a} = \vec{a} \\ \vec{b} &= (\vec{c} \times \vec{\bar{I}}) \cdot \vec{a} = \vec{c} \times \vec{a}. \end{aligned} \quad (\text{R.8})$$

In the mappings above,  $\vec{\bar{I}}$  is the identic or unit dyadic, corresponding to the unit matrix

$$\vec{\bar{I}} \cdot \vec{a} = \vec{a} \cdot \vec{\bar{I}} = \vec{a}. \quad (\text{R.9})$$

The identic dyadic can be presented by orthonormal basis vectors

$$\vec{\bar{I}} = \vec{e}_1 \vec{e}_1 + \vec{e}_2 \vec{e}_2 + \vec{e}_3 \vec{e}_3. \quad (\text{R.10})$$

The transpose of a dyadic is defined as

$$(\vec{a}\vec{b})_{\text{T}} = \vec{b}\vec{a}. \quad (\text{R.11})$$

If  $\vec{\bar{A}}_{\text{T}} = \vec{\bar{A}}$ , dyadic  $\vec{\bar{A}}$  is symmetric. If  $\vec{\bar{A}}_{\text{T}} = -\vec{\bar{A}}$ , dyadic  $\vec{\bar{A}}$  is antisymmetric. E.g.,  $\vec{a}\vec{b} + \vec{b}\vec{a}$ ,  $\vec{a}\vec{a} + \vec{b}\vec{b}$  and  $\vec{\bar{I}}$  are symmetric dyadics, and, e.g.,  $\vec{a}\vec{b} - \vec{b}\vec{a}$  is an antisymmetric dyadic. Because



$$\vec{a}\vec{b} = \frac{1}{2}((\vec{a}\vec{b} + \vec{b}\vec{a}) + (\vec{a}\vec{b} - \vec{b}\vec{a})), \quad (\text{R.12})$$

every dyadic can be presented as a sum of a symmetric and an antisymmetric dyadic.

As products between dyadics can be defined as the dot product

$$(\vec{a}\vec{b}) \cdot (\vec{c}\vec{d}) = (\vec{b} \cdot \vec{c})\vec{a}\vec{d} \quad (\text{R.13})$$

and the double dot product

$$(\vec{a}\vec{b}) : (\vec{c}\vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}). \quad (\text{R.14})$$

Other products can also be defined. In some references, the double dot product is defined in a different way so that there is one dot product between the outer vectors and another one between the inner vectors.

With the products between dyadics, we have following properties

$$\begin{aligned} \overline{\overline{A}} \cdot \left( \overline{\overline{B \cdot C}} \right) &= \left( \overline{\overline{A \cdot B}} \right) \cdot \overline{\overline{C}} \\ \overline{\overline{A \cdot B}} &\neq \overline{\overline{B \cdot A}} \\ \overline{\overline{A : B}} &= \overline{\overline{B : A}} = \overline{\overline{A_T : B_T}} \\ \left( \overline{\overline{A \cdot B}} \right)_T &= \overline{\overline{B_T \cdot A_T}}. \end{aligned} \quad (\text{R.15})$$

With the differential operators, we have, e.g., the following properties

$$\begin{aligned}
 \nabla(a\vec{b}) &= (\nabla a)\vec{b} + a\nabla\vec{b} \\
 \nabla \cdot (\vec{a}\vec{b}) &= (\nabla \cdot \vec{a})\vec{b} + \vec{a} \cdot \nabla\vec{b} \\
 \nabla \cdot (\vec{a}\vec{b} - \vec{b}\vec{a}) &= -\nabla \times (\vec{a} \times \vec{b}) \\
 (\nabla\vec{a})_{\text{T}} - \nabla\vec{a} &= (\nabla \times \vec{a}) \times \vec{I} \\
 \nabla \cdot \nabla\vec{a} &= \nabla^2\vec{a} \\
 \nabla \cdot (\nabla\vec{a})_{\text{T}} &= \nabla(\nabla \cdot \vec{a}) \\
 \nabla \cdot (\vec{a}\vec{A}) &= \nabla a \cdot \vec{A} + a\nabla \cdot \vec{A} \\
 \nabla \cdot (\vec{A} \cdot \vec{a}) &= \vec{a} \cdot (\nabla \cdot \vec{A}) + \vec{A} : \nabla\vec{a} \\
 \nabla \times \nabla\vec{a} &= 0,
 \end{aligned} \tag{R.16}$$

where the gradient of a vector is a dyadic, and in Cartesian coordinates as

$$\nabla\vec{a} = \vec{e}_x \frac{\partial\vec{a}}{\partial x} + \vec{e}_y \frac{\partial\vec{a}}{\partial y} + \vec{e}_z \frac{\partial\vec{a}}{\partial z}. \tag{R.17}$$



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|---|-------------------------------------|---|
| Author(s)<br>Seppo Uosukainen   |                                     |   |
| Title<br><b>Foundations of acoustic analogies</b>   |                                     |   |
| Abstract<br>This report presents the best-known acoustic analogies, and their equations are derived mathematically in detail to allow their applicability to be extended when necessary. In the acoustic analogies, the equations governing the flow-generated acoustic fields are rearranged in such a way that the field variable connections (wave operator part) are on the left-hand side and that which is supposed to form the source quantities for the acoustic field (source part) is on the right-hand side. Lighthill's analogy was originally developed for unbounded flows. The analogy assumes that, outside the source region, there is no static flow and the fluid is ideal. The refraction effects are not included in the wave operator. Powell's analogy is an approximate version of Lighthill's analogy. The Ffowcs Williams–Hawkings analogy is such an extension of Lighthill's analogy that, being based on the same starting point, it takes into account the effects of moving boundaries by equivalent Huygens sources. Curle's analogy is obtained from the Ffowcs Williams–Hawkings analogy by assuming that the boundaries are not moving. In Phillips' analogy, the effects of a moving medium are partially taken into account, and the refraction effects are included in the wave operator. The fluid outside the source region is assumed to be ideal. Lilley's analogy is based on the same starting point as Phillips' analogy, but all the 'propagation effects' occurring in a transversely sheared mean flow are inside the wave operator part of the equation. In Howe's analogy, the vorticity vector (in the form of Coriolis acceleration) and the entropy gradients are put in the source part of the equation, forming the main part of the sources; the compressibility of the medium is assumed to be constant and the viscous losses are assumed to vanish. In Doak's analogy, the compressibility of the medium does not need to be constant, the vorticity and the entropy gradients do not need to disappear outside the source region, and the viscous and thermal losses can be taken into account, somehow, inside and outside the source region. The four last-presented analogies assume that the medium is an ideal gas, so without modifications they cannot be applied to acoustic fields in liquids. |                                     |   |
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|--|--------------------------------|--|
| Tekijä(t)<br>Seppo Uosukainen  |                                |  |
| Nimeke<br><b>Akustisten analogioiden perusteet</b>   |                                |  |
| Tiivistelmä<br>Tunnetuimmat akustiset analogiat esitetään ja niitä hallitsevat yhtälöt johdetaan matemaattisesti yksityiskohtaisesti, jotta niiden sovellettavuutta olisi mahdollista tarvittaessa laajentaa. Akustisissa analogioissa virtausherätteisiä akustisia kenttiä hallitsevat yhtälöt järjestellään siten, että vasemmalla puolella ovat kenttämuuuttujayhteydet (aalto-operaattoriosia) ja oikealla puolella jotakin, jonka oletetaan muodostavan akustisen kentän lähdesuureet (lähdeosa). Lighthillin analogia on alun perin kehitetty rajoittamattomille virtauksille. Analogiassa oletetaan, että lähdealueen ulkopuolella ei ole staattista virtausta ja että fluidi on ideaalinen. Aalto-operaattori ei sisällä aallon taittumisen vaikutuksia. Powellin analogia on approksimaatio Lighthillin analogiasta. Ffowcs Williamsin–Hawkinsin analogia on Lighthillin analogian laajennus siten, että, perustuen samaan lähtökohtaan, se ottaa huomioon liikkuvien rajapintojen vaikutukset ekvivalenttisilla Huygensin lähteillä. Curlen analogia saadaan Ffowcs Williamsin–Hawkinsin analogiasta olettamalla rajapinnat liikkumattomiksi. Phillipsin analogiassa liikkuvan väliaineen vaikutukset otetaan osittain huomioon ja aallon taittumisen vaikutukset sisältyvät aalto-operaattoriin. Lähdealueen ulkopuolinen fluidi oletetaan ideaaliseksi. Lilleyin analogia perustuu samaan lähtökohtaan kuin Phillipsin analogia, mutta kaikki virtausten sekoittumisessa esiintyvät ”etenemisvaikutukset” ovat yhtälön aalto-operaattoripuolella. Howen analogiassa pyörteisyysvektori (Coriolis-kiihtyvyyden muodossa) ja entropiagradientti on sijoitettu yhtälön lähdepuolelle niiden muodostaessa pääosan lähteistä, väliaineen puristuvuuden oletetaan olevan vakio sekä viskoottisia häviöitä ei oteta huomioon. Doakin analogiassa puristuvuuden ei tarvitse olla vakio, pyörteisyys ja entropiagradienitien ei tarvitse hävitä lähdealueen ulkopuolella, ja viskoottiset ja lämpöhäviöt voidaan jollain lailla ottaa huomioon sekä lähdealueen sisä- että ulkopuolella. Neljässä viimeiseksi esitetystä analogiassa väliaine oletetaan ideaalikaasuksi, joten ilman modifikaatioita niitä ei voi soveltaa akustisiin kenttiin nesteissä. |                                |  |
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Acoustic analogies are used to govern the flow-generated acoustic fields. The best known of these are presented and their equations are derived. Lighthill's analogy is developed for unbounded flows with no static flow outside the source region and no refraction effects. Powell's analogy is an approximate version of Lighthill's analogy. The Ffowcs Williams-Hawkings analogy takes into account moving boundaries and Curle's analogy takes into account stationary boundaries. In Phillips' and Lilley's analogies, the effects of a moving medium and the refraction effects are included. In Howe's and Doak's analogies, the vorticity and the entropy gradients play an important role as sources. The four last analogies assume that the medium is an ideal gas, so without modifications they cannot be applied to acoustic fields in liquids.